Specially Structured Linear Programmes I: Transportation and Transhipment Problems

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A Typical Transportation Problem

- Inputs:
  - Sources with availability
  - Destinations with requirements
  - Unit cost of transportation from various sources to destinations
- Objective:
  - To determine schedule of transportation to minimise total transportation cost

Transportation Problem

Simplex Tableau: Optimal Solution

From ↓ To →		**Destinations**				**Supply**
		P	**Q**	**R**	**S**	
Source	**A**	12	10	12	13	500
	B	7	11	8	14	300
	C	6	16	11	7	200
Demand		180	150	350	320	1,000

Transportation Problem

- Number of sources = 3
- Number of destinations = 4
- There are 12 routes available, represented by 12 cells
- Aggregate demand = Aggregate supply = 1000 units
- This is a balanced transportation problem
- For a problem, if Aggregate demand ≠ Aggregate supply, the problem is called unbalanced transportation problem

LP Formulation of Transportation Problem

Min Z = $12\ x_{11} + 10\ x_{12} + 12\ x_{13} + 13\ x_{14} +$
$7\ x_{21} + 11\ x_{22} + 8\ x_{23} + 14\ x_{24} + 6\ x_{31} + 16x_{32}$
$+ 11\ x_{33} + 7\ x_{34}$

Subject to

$x_{11} + x_{12} + x_{13} + x_{14} = 500$
$x_{21} + x_{22} + x_{23} + x_{24} = 300$
$x_{31} + x_{32} + x_{33} + x_{34} = 200$
$x_{11} + x_{21} + x_{31} = 180$
$x_{12} + x_{22} + x_{32} = 150$
$x_{13} + x_{23} + x_{33} = 350$
$x_{14} + x_{24} + x_{34} = 320$
$x_{ij} \geq 0$ $i = 1, 2,3$ and $j = 1,2,3,4$

- There are 12 decision variables and 7 constraints. One of the constraints is redundant

Transportation Method

- Step 1
  - Balance the problem if AD and AS are unequal; place an M in the cost cell if some route is prohibited; and convert into equivalent min problem if it is a max problem
- Step 2
  - Obtain initial solution
- *Methods:*
  - **North-West Corner Rule**: considers only demand and availability
  - **Least Cost Method**: considers unit cost, demand and availability
  - **Vogel's Approximation Method**: considers cost differences, demand and availability

Initial Solution: NWC Method

TC=Rs. 10,220

From ↓ To →		**Destinations**				**Supply**
		P	**Q**	**R**	**S**	
Source	**A**	180 12	150 10	170 12	13	500
	B	7	11	180 8	120 14	300
	C	6	16	11	200 7	200
Demand		180	150	350	320	1,000

Allocation Sequence

1. Allocate 180 units on route AP
2. Move to cell AQ and allocate 150 units
3. Move to cell AR and allocate 170 units
4. Move to cell BR and allocate 180 units
5. Move to cell BS and allocate 120 units
6. Finally, allocate 200 units to cell CS

Initial Solution: Least Cost Method

TC=Rs. 9,620

From ↓ To →		Destinations				Supply
		P	Q	R	S	
Source	A	12	150 10	50 12	300 13	500
	B	7	11	300 8	14	300
	C	180 6	16	11	20 7	200
Demand		180	150	350	320	1,000

Allocation Sequence

1. Allocate 180 units on route AP
2. Select least cost cell CP, allocate 180 units
3. Delete column P, adjust supply at C to 20
4. Select cell CS (least cost among the undeleted cells) and allocate 20 units. Delete the row C and adjust demand at S as 300
5. Select cell BR, assign 300 units to it. Delete the row B and adjust demand at R to 50
6. Finally, Allocate 150 units to AQ, 50 to AR and 300 to AS

Initial Solution: VAM or Penalty Method

TC=Rs. 9,440

From ↓ **To** →		**Destinations**				**Supply**
		P	**Q**	**R**	**S**	
Source	**A**	12	150 10	230 12	120 13	500
	B	180 7	11	120 8	14	300
	C	6	16	11	200 7	200
Demand		180	150	350	320	1,000

- VAM involves obtaining and using cost differences
- For every row and every column, obtain the difference between two least cost values
- Select largest of the cost differences and choose the least cost cell in that row/column

Initial Solution: VAM or Penalty Method *(...continued)*

- Allocate the quantity equal to lower of the supply and demand values
- Delete the row or column that is satisfied
- Repeat the steps until all supply/demand requirements are met with
- Allocations are:

Step	**Cost Diff.**	**Cell**	**Qty.**	**Adjustment**
1	6	CS	200	Delete row C Revise col S 120
2	5	BP	180	Delete col P Revise row B 120
3	4	BR	120	Delete row B Revise col R 230
4	Allocate 150 units to AQ, 230 to AR and 120 to AS			

Transportation Method *(…continued)*

Step 3

Test of optimality: **MODI Method**

- First of all, check if the number of occupied cells = $m + n - 1$, where

  m = number of columns

  n = number of rows

- If yes, proceed to assign u_i values to the rows and v_j values to the columns, and calculate Δ_{ij} values for unoccupied cells
- Assign arbitrarily a value to *u* or *v.* You can assign any one u_i or v_j value. Let $u_1 = 0$
- Since $u_1 = 0$, select an occupied cell in row 1, find v_j value such that $u_i + v_j = c_{ij}$. With $u_1 = 0$, for AQ, $v_2 = 10$; for AR, $v_3 = 12$ and for AS, $v_4 = 13$

MODI Method *(…continued)*

- Using v_3 =12 choose BD and u_2 = -4, since 12 – 4 = 8 (cost)
- With u_2 = -4, choose BP, assign v_1 =11
- Finally, with v_4 = 13, and the occupied cell CS, assign u_3 = -6, as 13 – 6 = 7
- Having assigned u_i and v_j values, consider all unoccupied cells one by one, add up corresponding row and column (u_i and v_j) values and subtract there from the cost element, c_{ij}. Now, $\Delta_{ij} = u_i + v_j - c_{ij}$
- What do Δ_{ij} values indicate?
  - If all Δ_{ij} values are ≤ 0, it is optimal solution
  - If all Δ_{ij} values are negative, it is unique optimal solution
  - If some Δ_{ij} value/s = 0 while others are negative, there are multiple optimal solutions

Testing Optimality: MODI Method

	P	Q	R	S	SS	u_i
A	12	150 10	230 12	120 13	500	**0**
B	180 7	11	120 8	14	300	**-4**
C	6	16	11	120 7	200	**-6**
DD	180	150	350	320	1,000	
v_j	**11**	**10**	**12**	**13**		

- Δ_{11} = -1; Δ_{22} = -5; Δ_{24} = -5; Δ_{31} = -1; Δ_{32} = -12; and Δ_{33} = -5
- Since all values are negative, this solution is optimal and unique

Transportation Method *(...continued)*

Step 4

If some $\Delta_{ij} > 0$, the solution is not optimal. To improve a non-optimal solution:

a) Begin with cell having largest Δ_{ij}, draw a closed loop (or path):
   - Move alternately between rows and columns
   - Stop only at occupied cells
   - Start with a + sign in the cell of origin, place – and + signs alternately on cells on the path

b) Consider cells with '–' sign, choose the least quantity in them; add it to each cell with '+' sign and subtract it from each cell with '–' sign

c) Obtain revised solution and go back to step 3

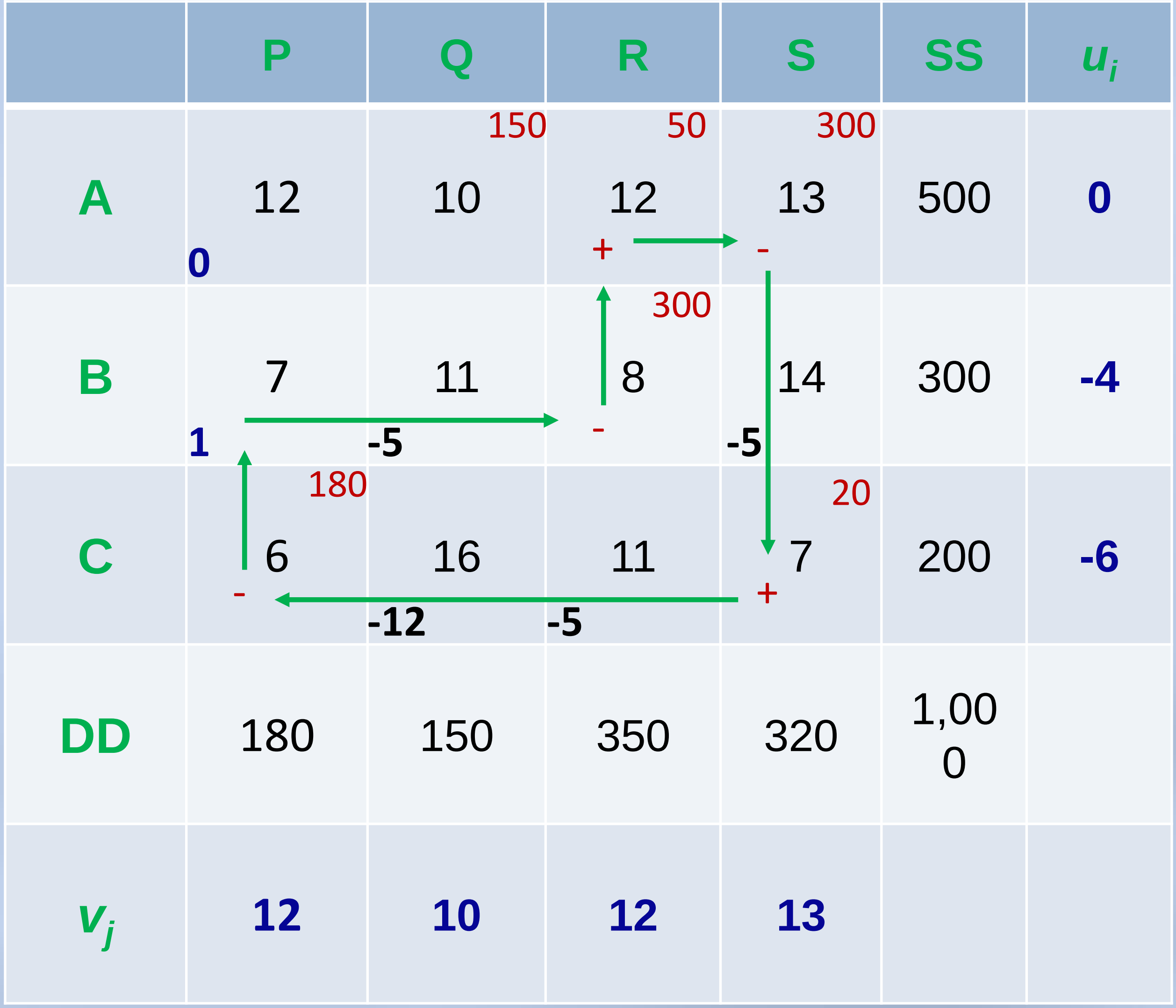
Closed Loop

A closed loop:

- Ends in the same cell where it begins
- Drawn through only occupied cells except the starting cell
- May be drawn clock-wise or anti clock-wise
- Always involves an even number of cells – involving equal number of cells with ‘–’ and ‘+’ signs
- Involves only horizontal and vertical movements
- Does never involve diagonal movement
- Can take any shape

Drawing of a Loop

Initial Solution - LCM

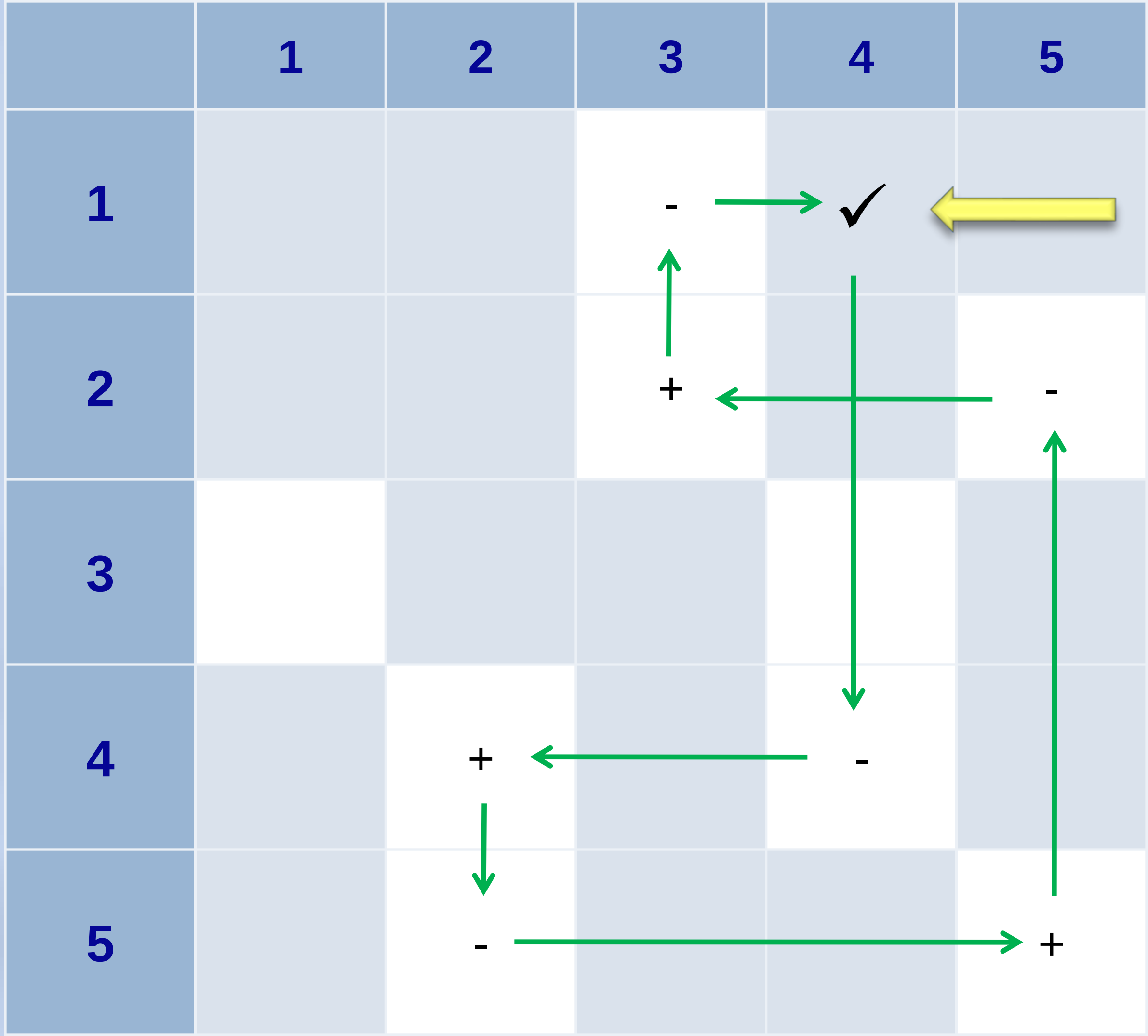


	P	Q	R	S	SS	u_i
A	12 0	150 10	50 12 +	300 13 -	500	**0**
B	7 1	11 -5	300 8 -	14 -5	300	**-4**
C	180 6 -	16 -12	11 -5	20 7 +	200	**-6**
DD	180	150	350	320	1,000	
v_j	**12**	**10**	**12**	**13**		

- **Loop:** BP – BR – AR – AS – CS – CP – BP
- Shall involve transferring 180 units

Drawing of a Loop
(...continued)

Occupied Cells → □



	1	2	3	4	5
1			-	✓	
2			+		-
3					
4		+		-	
5		-			+

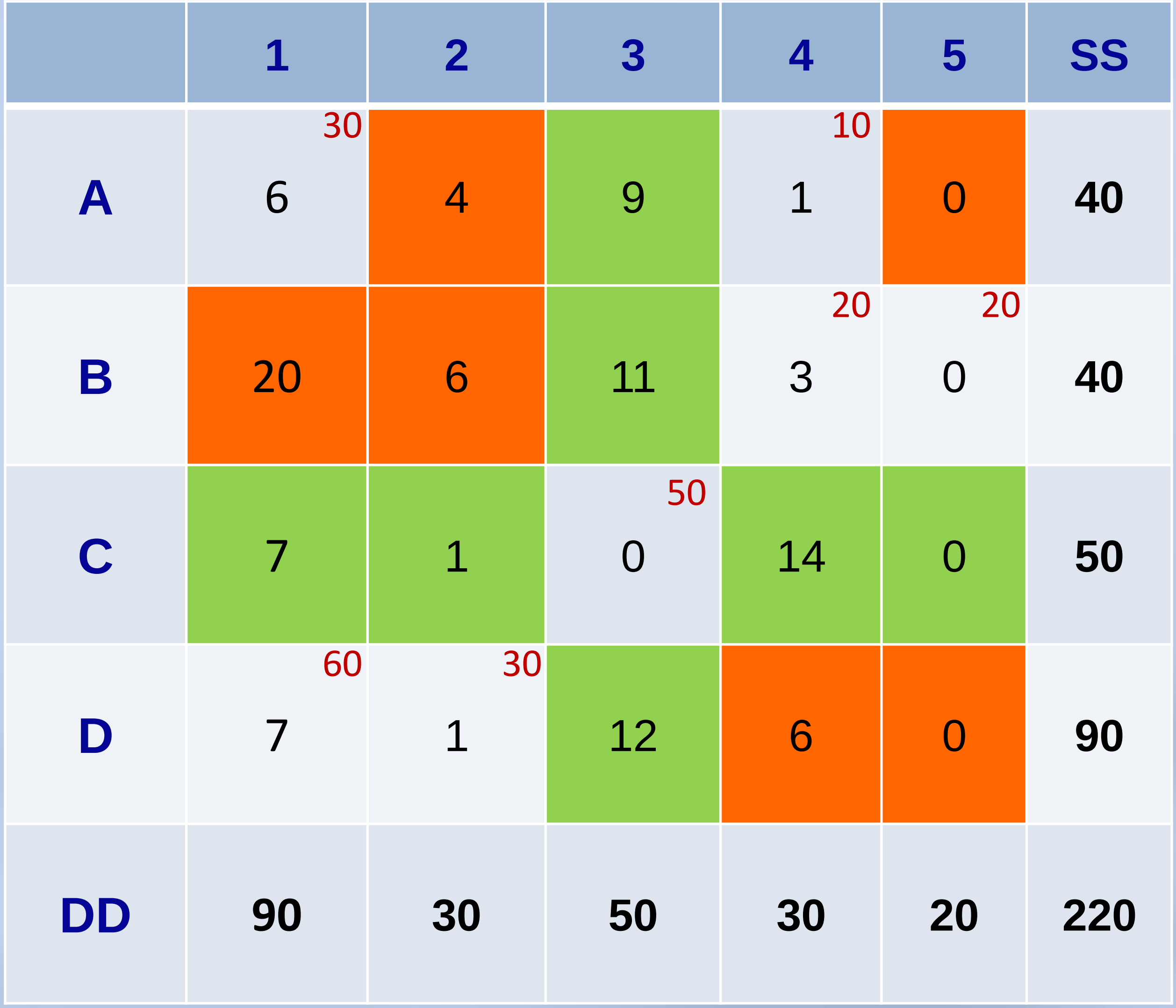
Start of Loop

- The loop can take any shape
- The loop can also cross over itself

Degeneracy

- Present when number of occupied cells is smaller than $m + n - 1$
- Prevents testing optimality of the solution
- Is removed by placing an infinitesimally small value ε in each of the required number of independent cells (An independent cell is one beginning from which a closed loop cannot be drawn)

Degenerate Solution



	1	2	3	4	5	SS
A	30 6	🟧 4	🟩 9	10 1	🟧 0	**40**
B	🟧 20	🟧 6	🟩 11	20 3	20 0	**40**
C	🟩 7	🟩 1	50 0	🟩 14	🟩 0	**50**
D	60 7	30 1	🟩 12	🟧 6	🟧 0	**90**
DD	**90**	**30**	**50**	**30**	**20**	**220**

🟧 Not independent cell

🟩 Independent cell

Dual of a Transportation Problem

- A transportation problem can be expressed as an LPP:
  - **Decision variables**: quantities shipped on various routes
  - **Constraints**: demand and availability values
  - **Objective function co-efficients**: cost elements
- As such, its dual can be written
- The u_i and v_j values in the optimal solution indicate optimal values of the dual variables

Dual of the Transportation Problem

Maximise $Z = 500u_1 + 300u_2 + 200u_3$
$+180v_1 + 150v_2 + 350v_3 + 320v_4$

Subject to

$u_1 + v_1 \le 12$
$u_1 + v_2 \le 10$
$u_1 + v_3 \le 12$
$u_1 + v_1 \le 13$
$u_2 + v_2 \le 7$
$u_2 + v_3 \le 11$
$u_2 + v_1 \le 8$
$u_2 + v_2 \le 14$
$u_3 + v_3 \le 6$
$u_3 + v_1 \le 16$
$u_3 + v_2 \le 11$
$u_3 + v_3 \le 7$

$u_1, u_2, u_3, v_1, v_2, v_3, v_4$: unrestricted in sign

Sensitivity Analysis

Deals with:

- Changes in cost element of a non-basic variable (an unoccupied cell)
- Changes in cost element of a basic variable (an occupied cell)
- Changes in a particular source supply and a particular destination demand by a given amount, *k*
- Effects of setting upper/lower limits on the values of x_{ij} variables

Production Planning Problem as TP

Key Data about an item:

- Can be produced in regular time (R) or overtime (O)
- Unit cost of production: R: Rs 6 and O: Rs 10
- Production capacity: R: 60 units and O: 20 units
- Carrying cost: Rs 3/unit/month
- Demand in Months 1, 2, 3 and 4: 50, 70, 85, and 75 units respectively

(continued...)

Production Planning Problem as TP

Origin	**Destination**					**Supply**
	1	**2**	**3**	**4**	**D**	
R	6	9	12	15	0	60
O	10	13	16	19	0	20
R	*M*	6	9	12	0	60
O	*M*	10	13	16	0	20
R	*M*	*M*	6	9	0	60
O	*M*	*M*	10	13	0	20
R	*M*	*M*	*M*	6	0	60
O	*M*	*M*	*M*	10	0	20
Demand	50	70	85	75	40	320

Transhipment Problem

- Allows for shipment from one source to another and from one destination to another
- An *m*-origin *n*-destination transportation problem becomes a *m*+*n* origin and *m*+*n* destination transhipment problem
- With minor modifications, problem is solved using transportation method
- Transhipment allows lowering total transportation cost

Multiple Choice Questions

- A transportation problem is balanced when

1. Total availability (TA) and Total Demand (TD) are equal, and number of sources is equal to number of destinations.
2. TA and TD are equal, irrespective of the number of sources and destinations.
3. Number of sources matches with number of destinations.
4. None of the routes is prohibited.

Multiple Choice Questions

- A solution to a transportation problem is degenerate when:

1. the number of unoccupied cells is less than $r + c - 1$ (r: rows, c: columns).
2. The number of unoccupied cells is equal to $r + c - 1$.
3. The number of occupied cells is at least $r + c - 1$.
4. The number of occupied cells is less than $r + c - 1$.

Multiple Choice Questions

- Mark the wrong statement:

1. An unbalanced transportation problem can be converted into a balanced transportation problem through the addition of an appropriate slack variable.
2. In North-West Corner Rule, first allocation is always made by beginning in the upper-left hand corner of the tableau.
3. The North-West Corner Rule provides a systematic but inefficient method of finding initial solution to a transportation problem.
4. It is necessary to make number of sources and destinations equal before applying N-W Corner Rule.

Multiple Choice Questions

- The cost elements of a row in a problem are: [32 18 24 18]. In applying VAM, what cost difference will be considered for this row?

1. 6
2. 0
3. 8
4. 14

Multiple Choice Questions

- Mark the wrong statement:

  1. The Least Cost Method always gives the least cost solution to a transportation problem.
  2. In the Penalty Method, a cost difference indicates the penalty of not using the least-cost route.
  3. In case there is a tie in the maximum penalty, either of them can be selected for making allocation.
  4. In comparison with the North-West Corner Rule, both, the Least-Cost Method and Vogel’s Approximation Method, are better because they provide “near optimal” initial solution.

Multiple Choice Questions

- To remove degeneracy, an ε is placed in an unoccupied cell which

1. has the least cost value.
2. has the largest cost value.
3. is an independent cell, beginning with which a closed path cannot be drawn.
4. is an independent cell, beginning with which a closed path can be drawn.

Multiple Choice Questions

- Which of the following conditions is not required to be satisfied by ε, used to remove degeneracy:

1. $k \times \varepsilon = 0$
2. $k - \varepsilon = k$
3. $\varepsilon - \varepsilon = 0$
4. $k \div \varepsilon = k$

Multiple Choice Questions

- Mark the correct statement:
  1. For drawing a closed path, movement has always to be clock-wise, never anti clock-wise.
  2. If all u_i and v_j values are known, one and only one closed path can be drawn starting with a given unoccupied cell.
  3. Any of the $r + c - 1$ number of occupied cells would allow determining whether a given solution is optimal or not.
  4. Degeneracy can arise only in initial solution to a problem.

Multiple Choice Questions

- Mark the wrong statement:
  1. The number of occupied cells involved in a closed path is always even.
  2. A closed path need not be a square/rectangular and can have any configuration.
  3. The maximum quantity, which can be reallocated in a closed path, is equal to the minimum quantity in the cells bearing negative sign.
  4. The $\Delta_{ij} = u_i + v_j - c_{ij}$ value of an unoccupied cell indicates the net change in cost of reallocating one unit through route involved.

Multiple Choice Questions

- Which of the following is not true for a transhipment problem?
  1. A transhipment problem allows for shipment of goods from one source to another as also from one destination point to another.
  2. An *m*-origin *n*-destination transportation problem, when converted into a transhipment problem, would become an $m \times n$-origin and an equal number of destinations problem.
  3. There is no real distinction between sources and destinations in a transhipment problem.
  4. A transhipment problem is likely to involve a lower cost than a transportation problem, in a given situation.