Extensions of Linear Programming:
Integer Programming and Goal Programming

Contents

Integer Programming

- **IPPs**: special type of LPPs where some or all variables are non-negative integers
- **Pure IPPs**: where all variables are required to be integers
- **Mixed IPPs**: where some variables are required to be integers
- **Zero-one Model**: an IPP where decision variables can take either of the two values: 0 or 1

Integer Programming *(...continued)*

Some examples of 0-1 IPP Model

- Assignment problem
- Knapsack problem
- Capital budgeting/investment problem
- Fixed charge problem
- Facility location problem

Assignment Problem as a 0 -1 IPP

$$\text{Min } Z = 45x_{11} + 40x_{12} + 51x_{13} + 67x_{14} + 57x_{21} + 42x_{22} + 63x_{23} + 55x_{24} + 49x_{31} + 52x_{32} + 48x_{33} + 64x_{34} + 41x_{41} + 45x_{42} + 60x_{43} + 55x_{44}$$

Subject to

$$x_{11} + x_{12} + x_{13} + x_{14} = 1$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 1$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 1$$
$$x_{41} + x_{42} + x_{43} + x_{44} = 1$$
$$x_{11} + x_{21} + x_{31} + x_{41} = 1$$
$$x_{12} + x_{22} + x_{32} + x_{42} = 1$$
$$x_{13} + x_{23} + x_{33} + x_{43} = 1$$
$$x_{14} + x_{24} + x_{34} + x_{44} = 1$$

$x_{ij} = 1$ if assignment is made

$x_{ij} = 0$ if assignment is not made

Integer Programming

Example 7.4 Data

Max $Z = 20{,}000x_1 + 28{,}000x_2 + 18{,}500x_3 + 27{,}500x_4 + 31{,}000x_5$

St $12{,}000x_1 + 14{,}000x_2 + 7{,}000x_3 + 13{,}000x_4 + 16{,}000x_5 \leq 48{,}000$

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 3$$

$$-x_2 + x_3 \leq 0$$

$$x_4 + x_5 \leq 1$$

$$x_1 , x_2 , x_3 , x_4 , x_5 = 0$$

if investment is not made, And

=1 if investment is made

Solution to IPPs

Cutting Plane Algorithm

- Drop the integer requirement and solve as LP relaxation

- If optimal solution has all-integer values, the problem is solved while if not, introduce a cut, called Gomory cut

- A cut is introduced using a constraint in the optimal solution that has a fractional RHS (b_i) value

- Cuts are introduced until all-integer solution is obtained

Properties of Gomory Cut

- The current optimal solution to LP relaxation does not satisfy the cut – Gomory's cut has the effect of removing optimal solution to the LPP

- Any feasible point for the IPP satisfies the cut as the cut does not remove any feasible solution of the original IPP

Introducing Gomory's Cut

- To introduce the cut, rewrite constraint by separating each of the co-efficients into two parts: an integer and a fraction
- The fraction is required to be non-negative
- In case of no fractional part, use 0/1 as the dummy fraction
- Put all integer terms separately from the fractional ones and then drop the integer terms on the LHS and change the constraint to ≥ form

Introducing Gomory's Cut

Example 1: Consider the following linear equality constraint:

$4x_1 + 1\ 1/4x_2 + 2/3x_3 = 5\ 2/3$

where x_1 , x_2 and x_3 are non-negative integers

- Rewrite it as follows:

$(4 + 0/1)x_1 + (1 + 1/4)x_2 + (0 + 2/3)x_3 = 5 + 2/3$

OR

$0/1x_1 + 1/4x_2 + 2/3x_3 + (4x_1 + x_2 + 0x_3 - 5) = 2/3$

- Dropping all integer terms,

$0x_1 + 1/4x_2 + 2/3x_3 \geq 2/3$

Introducing Gomory's Cut

Example 2:

$4x_1 - 1\ 1/4x_2 + 2/3x_3 = 5\ 2/3$

where x_1 , x_2 and x_3 are non-negative integers

- To develop the cut, write

$(4 + 0/1)x_1 + (-2 + 3/4)x_2 + (0 + 2/3)x_3 = 5 + 2/3$

Remember the fraction has to be non-negative

OR

$0/1x_1+3/4x_2+2/3x_3+(4x_1 - 2x_2 + 0x_3 - 5) = 2/3$

- Thus,

$0x_1+3/4x_2+2/3x_3 \geq 2/3$

Introducing Gomory Constraints : Rules

- Change the co-efficients of the constraint including RHS as:

Rule	Original Co-efficient	Gomory Cut
1	Integer (Positive or Negative)	Zero
2	Positive Fraction, e.g. 3/5	Unchanged, 3/5
3	Negative fraction, e.g. -3/5	Positive complement, 2/5

- Change = sign to ≥

Solution to IPPs

Branch and Bound Method

- Drop the integer requirement and solve as LP relaxation
- Not a particular algorithm, but rather a generalised method
- Used generally in combinatorial problems
- Involves dividing solutions into two parts – one that most probably contains optimal solution and second that does not
- Examine further the first part to get optimal solution

Goal Programming

- Deals with multiple-objective situations
- Multiple goals are usually conflicting in nature
- Uses the concept of penalties in the format of linear programming
- A value of each goal is specified and deviational variables are used

Goal Programming

Non-preemptive Goal Programming

- Assumes the decision-maker has a linear utility function with respect to the objectives
- Differential weights are used to various goals in line with their relative significance

Preemptive Goal Programming

- Uses prioritised goals
- When optimal solution with respect to the higher priority goal is obtained only then the next priority level is considered

Multiple Choice Questions

- Mark the correct statement about integer programming problems (IPPs):
  1. Pure IPPs are those problems in which all the variables are non-negative.
  2. The 0-1 IPPs are those in which all variables are either 0 or all are equal to 1.
  3. Mixed IPPs are that where decision variables can take integer values only but the slack/surplus variables can take fractional values as well.
  4. In real life, no variable can assume fractional values. Hence we should always use IPPs.

Multiple Choice Questions

- Consider the following problem:

  Max. $z = 28x_1 + 32x_2$, subject to $5x1 + 3x_2 \leq 23$, $4x_1 + 7x_2 \leq 33$, and $x_1 \geq 0$, $x_2 \geq 0$. This problem is:

1. A pure IPP.
2. A 0-1 IPP.
3. A mixed IPP.
4. Not an IPP.

Multiple Choice Questions

- Mark the wrong statement:

1. An IPP where the values of the decision variables are limited to two logical categories, like yes or no, are handled as 0-1 problems.

2. An assignment problem can be expressed as a 0-1 IPP.

3. An IPP that has no constraint is known as knapsack problem.

4. For a problem involving three variables x_1, x_2, $x_3 \geq 0$, and x_1, x_3 integer, is a mixed IPP.

Multiple Choice Questions

- For a capital budgeting problem with five investment projects, state which of the following is incorrect (in each case $x_1, x_2, x_3, x_4, x_5 = 0$ or 1):
  1. No more then three investments be made: $x_{1\,+}\ x_2 + x_3 + x_4 + x_5 \leq 3$
  2. Invest in proposal III if invested in proposal V: $x_2 \leq x_3$
  3. Do not invest in 5, invest in three: $x_3 + x_5 \leq 1$
  4. Total budget is Rs. 72,000 and cash outflow (in '000 Rs.) is 28, 32, 18, 38 and 19: $x_{1\,+}\ x_2 + x_3 + x_4 + x_5 \leq 3$

Multiple Choice Questions

- Mark the correct statement:

1. A facility location problem can be formulated and solved as a 0-1 IPP.
2. Problems involving piece-wise linear functions can be modelled as mixed linear programming problems.
3. Solution to an IPP can be obtained by first solving the problem as an LPP then rounding off the fractional values.
4. If the optimal solution to an LPP has all integer values, it may or may not be an optimal integer solution.

Multiple Choice Questions

- Mark the wrong statement:
  1. To solve an IPP using cutting plane algorithm, the integer requirements are dropped in the first instance to obtain LP relaxation.
  2. A cut is formed by choosing a row in the optimal tableau that corresponds to a non-integer variable.
  3. A constraint picked from the optimal tableau is: $0x_1 + x_2 + ½\, S_1 – 1/3\, S_2 = 7/2$. With S_3 being a slack variable introduced, the cut would be: $-1/2\, S_1 – 2/3\, S_2 + S_3 = -1/2$
  4. The optimal solution to LPP satisfies the cut that is introduced on the basis of it.

Multiple Choice Questions

- In cutting plane algorithm, each cut which is made involves the introduction of:

1. An = constraint.
2. An artificial variable.
3. a $\leq$ constraint.
4. a $\geq$ constraint.

Multiple Choice Questions

- Which of the following effects does the addition of a Gomory have? (i) adding a new variable to the tableau; (ii) elimination of non-integer solutions from the feasibility region; (iii) making the previous optimal solution infeasible by eliminating that part of the feasible region which contained that solution.

1. (i) only
2. (i) and (ii) only
3. (i) and (iii) only
4. all the above

Multiple Choice Questions

- Mark the incorrect statement about Branch and Bound method.

1. It is not a particular method and is used differently in different kinds of problems.
2. It is generally used in combinatorial problems.
3. It divides the feasible region into smaller parts by the process of branching.
4. It can be used for solving any kind of programming problem.

Multiple Choice Questions

- Mark the wrong statement:

1. Goal programming deals with problems with multiple goals.

2. Goal programming realizes that goals may be under-achieved, over-achieved or met exactly.

3. The inequalities or equalities representing goal constraints are flexible.

4. The initial tableau of a goal programming problem should never have a variable in the basis which is an under-achievement variable.

Multiple Choice Questions

- Mark the wrong statement:
  1. A travelling salesman problem can be solved using Branch and Bound method.
  2. An assignment problem can be formulated as a 0-1 IPP and solved using Branch and Bound method.
  3. The Branch and Bound method terminates when the upper and lower bound become identical and the solution is that single value.
  4. The Branch and Bound method can never reveal multiple optimal solutions to a problem, if they exist.

Multiple Choice Questions

- If two deviational variables, d- for under-achievement and d+ for over-achievement, are introduced in a goal constraint, then which of the following would not hold:

1. Each of them can be positive or zero.
2. Either d- or d+ is zero.
3. Both are non-zero.
4. Both are equal to zero.

Multiple Choice Questions

- Mark the wrong statement:

1. A ‘lower’ one-sided goal sets a lower limit that we do not want to fall under.
2. A two-sided goal sets a specific target missing which from either side is not desired.
3. In goal programming, an attempt is made to minimise deviations from targets.
4. In using goal programming, one has to specify clearly the relative importance of the various goals involved by assigning weights to them.

Multiple Choice Questions

- Mark the wrong statement:

1. 'Non pre-emptive goal programming assumes that the decision-maker has a linear utility function with respect to the objectives.
2. Deviations for various goals may be given penalty weights in accordance with the relative significance of the objectives.
3. The penalty weights measure the marginal rate of substitution between the objectives.
4. A goal-programming problem cannot have multiple optimal solutions.

Multiple Choice Questions

- Mark the wrong statement:

1. In pre-emptive goal programming, the goals are ranked from least important (goal 1) to most important (goal n), with objective function co-efficients P_i , such that $P_1 >>> P_2 >>> P_3 .... >>> P_n$.
2. Existence of multiple Δ_j rows indicates a prioritised goal-programming problem.
3. A lower priority is never sought to be achieved at the expense higher-priority goal.
4. The coefficients, P_i's are not assigned any actual values.