Integer Linear Programming

- Types of Integer Linear Programming Models
- Graphical and Computer Solutions for an All-Integer Linear Program
- Applications Involving 0-1 Variables
- Modeling Flexibility Provided by 0-1 Variables

Types of Integer Programming Models

- An LP in which all the variables are restricted to be integers is called an all-integer linear program (ILP).
- The LP that results from dropping the integer requirements is called the LP Relaxation of the ILP.
- If only a subset of the variables are restricted to be integers, the problem is called a mixed-integer linear program (MILP).
- Binary variables are variables whose values are restricted to be 0 or 1. If all variables are restricted to be 0 or 1, the problem is called a 0-1 or binary integer linear program.

Example: All-Integer LP

- Consider the following all-integer linear program:

$$\begin{aligned} \text{Max} \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & 3x_1 + x_2 \leq 9 \\ & x_1 + 3x_2 \leq 7 \\ & -x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \text{ and integer} \end{aligned}$$

Example: All-Integer LP

- LP Relaxation

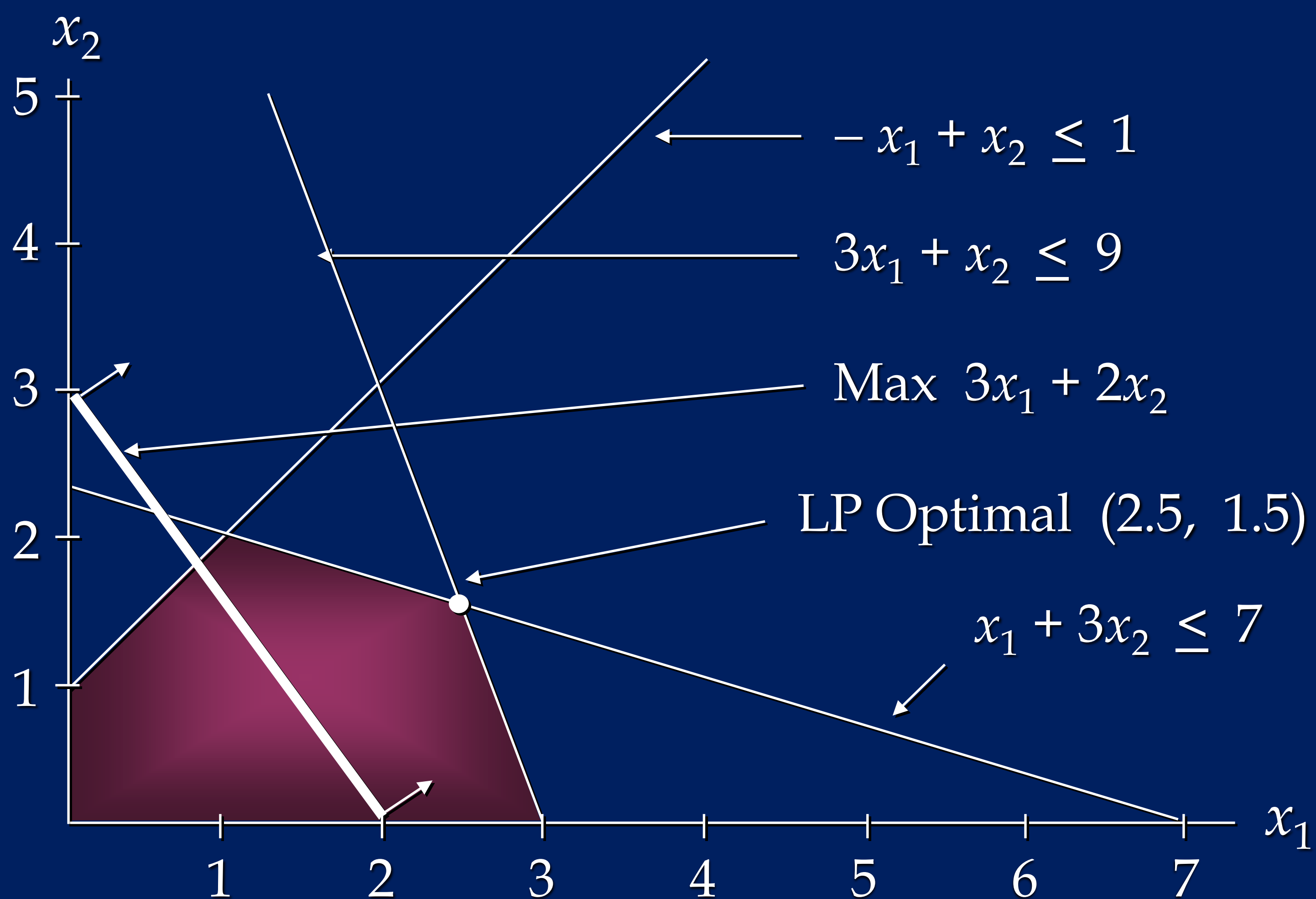
    Solving the problem as a linear program ignoring the integer constraints, the optimal solution to the linear program gives fractional values for both x_1 and x_2. From the graph on the next slide, we see that the optimal solution to the linear program is:

$$x_1 = 2.5, \quad x_2 = 1.5,$$

$$\text{Max } 3x_1 + 2x_2 = 10.5$$

Example: All-Integer LP

- LP Relaxation



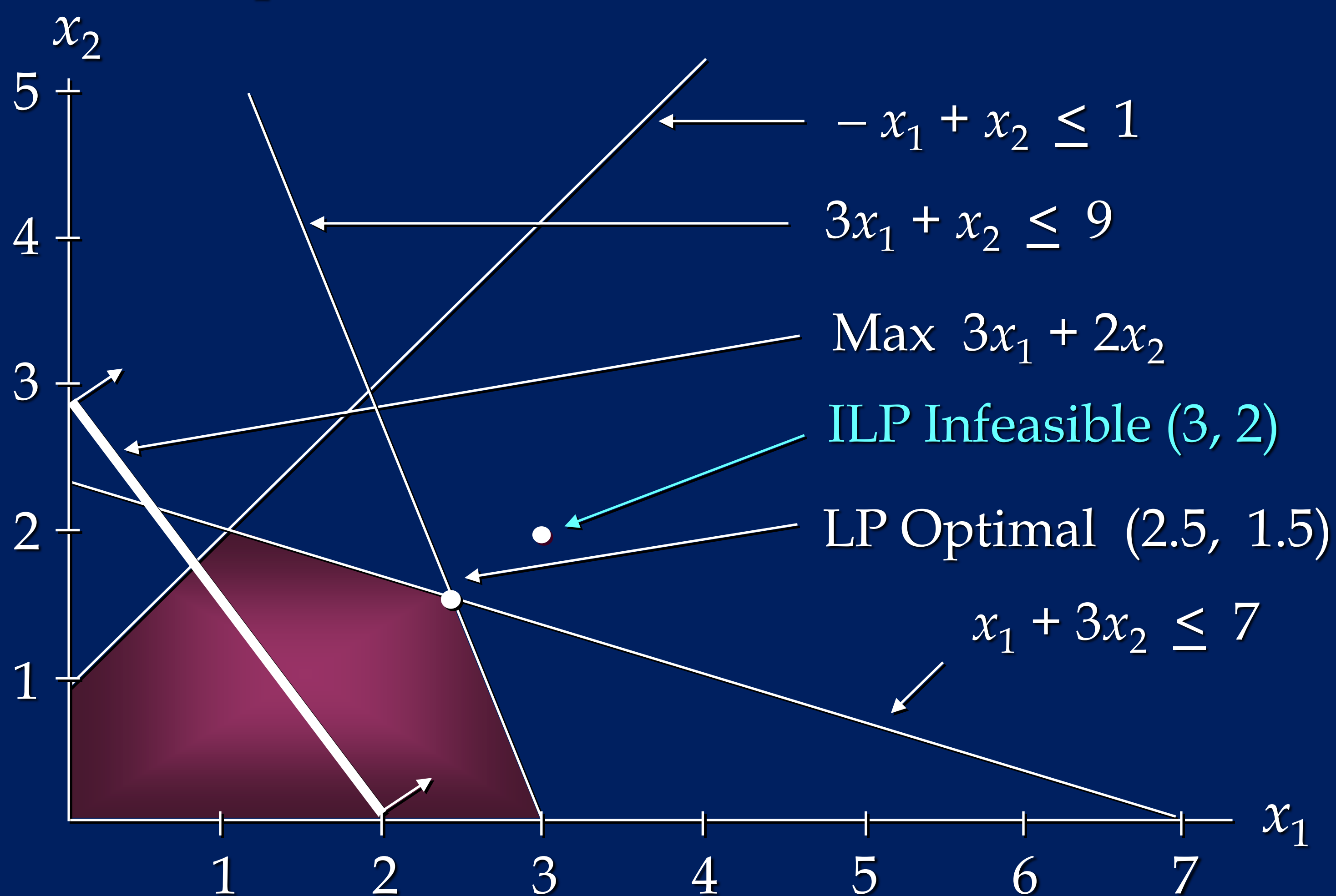
Example: All-Integer LP

- Rounding Up

  If we round up the fractional solution (x_1 = 2.5, x_2 = 1.5) to the LP relaxation problem, we get x_1 = 3 and x_2 = 2. From the graph on the next slide, we see that this point lies outside the feasible region, making this solution infeasible.

Example: All-Integer LP

- Rounded Up Solution



Example: All-Integer LP

- Rounding Down

By rounding the optimal solution down to $x_1 = 2$, $x_2 = 1$, we see that this solution indeed is an integer solution within the feasible region, and substituting in the objective function, it gives $3x_1 + 2x_2 = 8$.

We have found a feasible all-integer solution, but have we found the OPTIMAL all-integer solution?

The answer is NO! The optimal solution is $x_1 = 3$ and $x_2 = 0$ giving $3x_1 + 2x_2 = 9$, as evidenced in the next two slides.

Example: All-Integer LP

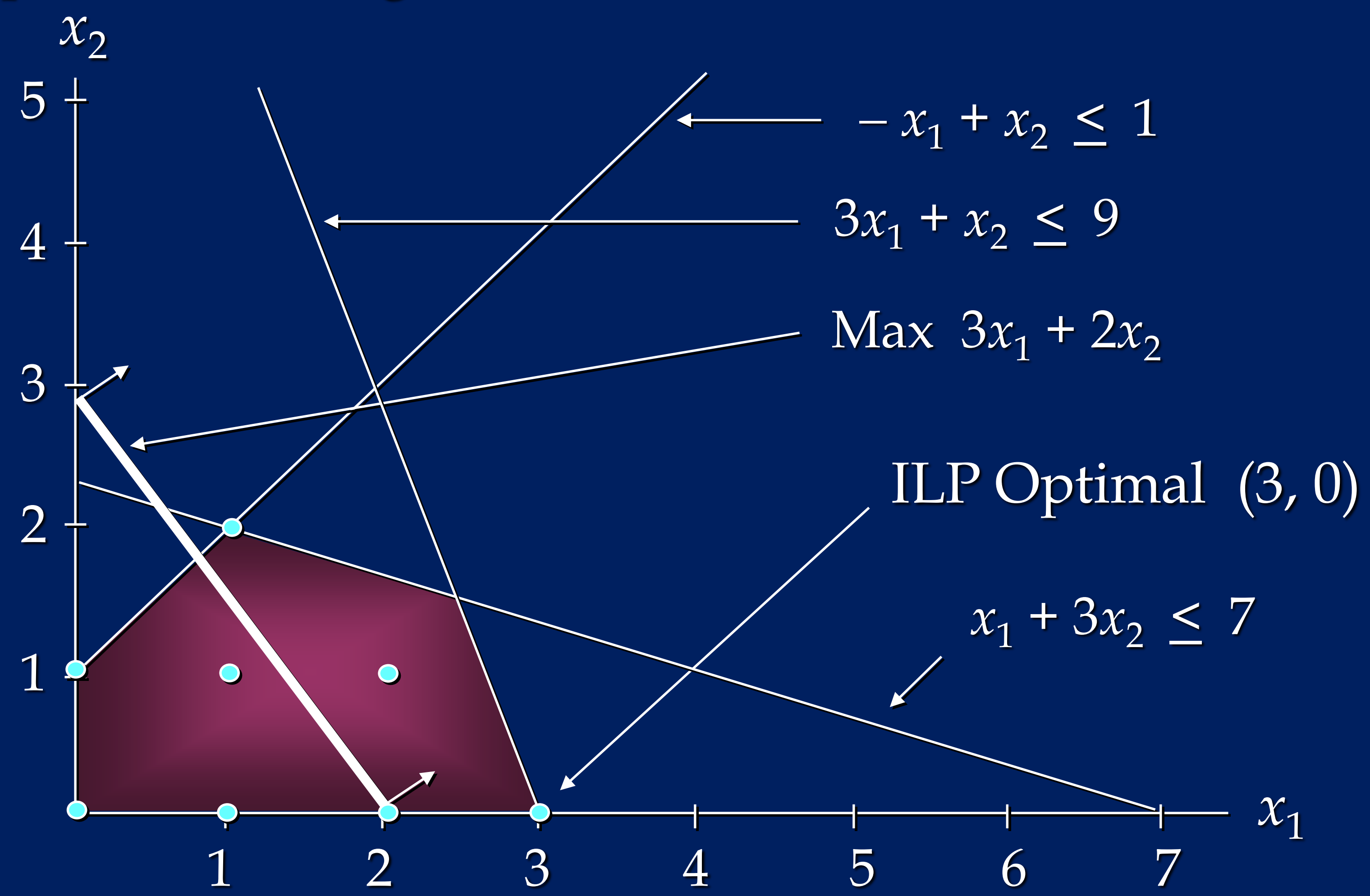
- Complete Enumeration of Feasible ILP Solutions

  There are eight feasible integer solutions to this problem:

	x_1	x_2	$3x_1 + 2x_2$	
1.	0	0	0	
2.	1	0	3	
3.	2	0	6	
4.	3	0	9	← optimal solution
5.	0	1	2	
6.	1	1	5	
7.	2	1	8	
8.	1	2	7	

Example: All-Integer LP

- Optimal All-Integer Solution



Example: Capital Budgeting

The Ice-Cold Refrigerator Company is considering investing in several projects that have varying capital requirements over the next four years. Faced with limited capital each year, management would like to select the most profitable projects. The estimated net present value for each project, the capital requirements, and the available capital over the four-year period are shown on the next slide.

Example: Capital Budgeting

- Problem Data

	Project				
	Plant Expansion	Warehouse Expansion	New Machinery	New Product Research	Total Capital Available
Present Value	$90,000	$40,000	$10,000	$37,000	
Year 1 Cap Rqmt	$15,000	$10,000	$10,000	$15,000	$40,000
Year 2 Cap Rqmt	$20,000	$15,000		$10,000	$50,000
Year 3 Cap Rqmt	$20,000	$20,000		$10,000	$40,000
Year 4 Cap Rqmt	$15,000	$ 5,000	$ 4,000	$10,000	$35,000

Example: Capital Budgeting

- Decision Variables

The four 0-1 decision variables are as follows:

P = 1 if the plant expansion project is accepted;
0 if rejected
W = 1 if the warehouse expansion project is accepted;
0 if rejected
M = 1 if the new machinery project is accepted;
0 if rejected
R = 1 if the new product research project is accepted;
0 if rejected

Example: Capital Budgeting

- Problem Formulation

Max $90P + 40W + 10M + 37R$

s.t. $15P + 10W + 10M + 15R \leq 40$ (Yr. 1 capital avail.)

$20P + 15W + 10R \leq 50$ (Yr. 2 capital avail.

$20P + 20W + 10R \leq 40$ (Yr. 3 capital avail.)

$15P + 5W + 4M + 10R \leq 35$ (Yr. 4 capital avail.)

$P, W, M, R = 0, 1$

Example: Capital Budgeting

- Optimal Solution

  P = 1, *W* = 1, *M* = 1, *R* = 0.
  Total estimated net present value = $140,000

  The company should fund the plant expansion, the warehouse expansion, and the new machinery projects. The new product research project should be put on hold unless additional capital funds become available.

  The company will have $5,000 remaining in year 1, $15,000 remaining in year 2, and $11,000 remaining in year 4. Additional capital funds of $10,000 in year 1 and $10,000 in year 3 will be needed to fund the new product research project.

Example: Fixed Cost

Three raw materials are used to produce 3 products: a fuel additive, a solvent base, and a carpet cleaning fluid. The profit contributions are $40 per ton for the fuel additive, $30 per ton for the solvent base, and $50 per ton for the carpet cleaning fluid.

Each ton of fuel additive is a blend of 0.4 tons of material 1 and 0.6 tons of material 3. Each ton of solvent base requires 0.5 tons of material 1, 0.2 tons of material 2, and 0.3 tons of material 3. Each ton of carpet cleaning fluid is a blend of 0.6 tons of material 1, 0.1 tons of material 2, and 0.3 tons of material 3.

Example: Fixed Cost

RMC has 20 tons of material 1, 5 tons of material 2, and 21 tons of material 3, and is interested in determining the optimal production quantities for the upcoming planning period.

There is a fixed cost for production setup of the products, as well as a maximum production quantity for each of the three products.

Product	Setup Cost	Maximum Production
Fuel additive	$200	50 tons
Solvent base	$ 50	25 tons
Cleaning fluid	$400	40 tons

Example: Fixed Cost

- Decision Variables

F = tons of fuel additive produced

S = tons of solvent base produced

C = tons of carpet cleaning fluid produced

SF = 1 if the fuel additive is produced; 0 if not

SS = 1 if the solvent base is produced; 0 if not

SC = 1 if the cleaning fluid is produced; 0 if not

Example: Fixed Cost

- Problem Formulation

$$\text{Max } 40F + 30S + 50C - 200SF - 50SS - 400SC$$

$$
\begin{array}{llr}
\text{s.t.} & 0.4F + 0.5S + 0.6C \leq 20 & \text{(Mat'l. 1)} \\
 & 0.2S + 0.1C \leq 5 & \text{(Mat'l. 2)} \\
 & 0.6F + 0.3S + 0.3C \leq 21 & \text{(Mat'l. 3)} \\
 & F - 50SF \leq 0 & \text{(Max.F)} \\
 & S - 25SS \leq 0 & \text{(Max. S)} \\
 & C - 50SF \leq 0 & \text{(Max. C)} \\
 & F, S, C \geq 0; \ SF, SS, SC = 0, 1 &
\end{array}
$$

Example: Fixed Cost

- Optimal Solution

Produce 25 tons of fuel additive.
Produce 20 tons of solvent base.
Produce 0 tons of cleaning fluid.

The value of the objective function after deducting the setup cost is $1350. The setup cost for the fuel additive and the solvent base is $200 + $50 = $250.

The optimal solution shows *SC* = 0, which indicates that the more expensive $400 setup cost for the carpet cleaning fluid should be avoided.

Example: Supply Chain Design

The Martin-Beck Company operates a plant in St. Louis with an annual capacity of 30,000 units. Product is shipped to regional distribution centers located in Boston, Atlanta, and Houston. Because of an anticipated increase in demand, Martin-Beck plans to increase capacity by constructing a new plant in one or more of the following cities: Detroit, Toledo, Denver, or Kansas City.

Example: Supply Chain Design

The estimated annual fixed cost and the annual capacity for the four proposed plants are as follows:

Proposed Plant	Annual Fixed Cost	Annual Capacity
Detroit	$175,000	10,000
Toledo	$300,000	20,000
Denver	$375,000	30,000
Kansas City	$500,000	40,000

Example: Supply Chain Design

The company's long-range planning group developed forecasts of the anticipated annual demand at the distribution centers as follows:

Distribution Center	Annual Demand
Boston	30,000
Atlanta	20,000
Houston	20,000

Example: Supply Chain Design

The shipping cost per unit from each plant to each distribution center is shown below.

	Distribution Centers		
Plant Site	Boston	Atlanta	Houston
Detroit	5	2	3
Toledo	4	3	4
Denver	9	7	5
Kansas City	10	4	2
St. Louis	8	4	3

Example: Supply Chain Design

- Decision Variables

y_1 = 1 if a plant is constructed in Detroit; 0 if not

y_2 = 1 if a plant is constructed in Toledo; 0 if not

y_3 = 1 if a plant is constructed in Denver; 0 if not

y_4 = 1 if a plant is constructed in Kansas City; 0 if not

x_{ij} = the units shipped (in 1000s) from plant i to distribution center j , with i = 1, 2, 3, 4, 5 and j = 1, 2, 3

Example: Supply Chain Design

- Problem Formulation

$$
\begin{aligned}
\text{Min}\quad & 5x_{11} + 2x_{12} + 3x_{13} + 4x_{21} + 3x_{22} + 4x_{23} + 9x_{31} + 7x_{32} + 5x_{33} + 10x_{41} + 4x_{42} \\
& + 2x_{43} + 8x_{51} + 4x_{52} + 3x_{53} + 175y_1 + 300y_2 + 375y_3 + 500y_4
\end{aligned}
$$

s.t.

$$
\begin{array}{llll}
x_{11} + x_{12} + x_{13} - 10y_1 & \leq & 0 & \text{Detroit capacity} \\
x_{21} + x_{22} + x_{23} - 20y_2 & \leq & 0 & \text{Toledo capacity} \\
x_{31} + x_{32} + x_{33} - 30y_3 & \leq & 0 & \text{Denver capacity} \\
x_{41} + x_{42} + x_{43} - 40y_4 & \leq & 0 & \text{Kansas City capacity} \\
x_{51} + x_{52} + x_{53} & \leq & 30 & \text{St. Louis capacity} \\
x_{11} + x_{21} + x_{31} + x_{41} + x_{51} & = & 30 & \text{Boston demand} \\
x_{12} + x_{22} + x_{32} + x_{42} + x_{52} & = & 20 & \text{Atlanta demand} \\
x_{13} + x_{23} + x_{33} + x_{43} + x_{53} & = & 20 & \text{Houston demand}
\end{array}
$$

$x_{ij} \geq$ for all i and j; $y_1, y_2, y_3, y_4 = 0, 1$

Example: Distribution System Design

- Optimal Solution

Construct plant in Kansas City ($y_4 = 1$).

Ship 20,000 units: Kansas City to Atlanta ($x_{42} = 20$),
Ship 20,000 units: Kansas City to Houston ($x_{43} = 20$),
Ship 30,000 units: St. Louis to Boston ($x_{51} = 30$).

Total cost: $860,000 including fixed cost of $500,000.

Example: Bank Location

The long-range planning department for the Ohio Trust Company is considering expanding its operation into a 20-county region in northeastern Ohio. Ohio Trust does not have, at this time, a principal place of business in any of the 20 counties.

According to the banking laws in Ohio, if a bank establishes a principal place of business (PPB) in any county, branch banks can be established in that county and in any adjacent county. To establish a new PPB, Ohio Trust must either obtain approval for a new bank from the state's superintendent of banks or purchase an existing bank.

Example: Bank Location

The 20 counties in the region and adjacent counties are listed on the next slide. For example, Ashtabula County is adjacent to Lake, Geauga, and Trumbull counties; Lake County is adjacent to Ashtabula, Cuyahoga, and Geauga counties; and so on.

As an initial step in its planning, Ohio Trust would like to determine the minimum number of PPBs necessary to do business throughout the 20-county region. A 0-1 integer programming model can be used to solve this **location problem** for Ohio Trust.

Example: Bank Location

Counties Under Consideration	Adjacent Counties (by Number)
1. Ashtabula	2, 12, 16
2. Lake	1, 3, 12
3. Cuyahoga	2, 4, 9, 10, 12, 13
4. Lorain	3, 5, 7, 9
5. Huron	4, 6, 7
6. Richland	5, 7, 17
7. Ashland	4, 5, 6, 8, 9, 17, 18
8. Wayne	7, 9, 10, 11, 18
9. Medina	3, 4, 7, 8, 10
10. Summit	3, 8, 9, 11, 12, 13
11. Stark	8, 10, 13, 14, 15, 18, 19, 20
12. Geauga	1, 2, 3, 10, 13, 16
13. Portage	3, 10, 11, 12, 15, 16
14. Columbiana	11, 15, 20
15. Mahoning	11, 13, 14, 16
16. Trumbull	1, 12, 13, 15
17. Knox	6, 7, 18
18. Holmes	7, 8, 11, 17, 19
19. Tuscarawas	11, 18, 20
20. Carroll	11, 14, 19

Example: Bank Location

- Decision Variables

  x_i = 1 if a PBB is established in county i; 0 otherwise

- Problem Formulation

$$
\begin{array}{llr}
\text{Min} & x_1 + x_2 + \cdots + x_{20} & \\
\text{s.t.} & & \\
& x_1 + x_2 + x_{12} + x_{16} \geq 1 & \text{Ashtabula} \\
& x_1 + x_2 + x_3 + x_{12} \geq 1 & \text{Lake} \\
& \cdot & \cdot \\
& \cdot & \cdot \\
& \cdot & \cdot \\
& x_{11} + x_{14} + x_{19} + x_{20} \geq 1 & \text{Carroll} \\
& x_i = 0, 1 \quad i = 1, 2, \ldots, 20 &
\end{array}
$$

Example: Bank Location

- Optimal Solution

For this 20-variable, 20-constraint problem:

Establish PPBs in Ashland, Stark, Geauga counties.

(With PPBs in these three counties, Ohio Trust can place branch banks in all 20 counties.)

All other decision variables have an optimal value of zero, indicating that a PPB should not be placed in these counties.

Example: Product Design & Market Share

Market Pulse Research has conducted a study for Lucas Furniture on some designs for a new commercial office desk. Three attributes were found to be most influential in determining which desk is most desirable: number of file drawers, the presence or absence of pullout writing boards, and simulated wood or solid color finish. Listed on the next slide are the part-worths for each level of each attribute provided by a sample of 7 potential Lucas customers.

Example: Product Design & Market Share

- Part-Worths

	File Drawer			Pullout Writing Boards		Finish	
Consumer	0	1	2	Present	Absent	Simulated Wood	Solid Color
1	5	26	20	18	11	17	10
2	18	11	5	12	16	15	26
3	4	16	22	7	13	11	19
4	12	8	4	18	9	22	14
5	19	9	3	4	14	30	19
6	6	15	21	8	17	20	11
7	9	6	3	13	5	16	28

Example: Product Design & Market Share

Suppose the overall utility (sum of part-worths) of the current favorite commercial office desk is 50 for each customer. What is the product design that will maximize the share of choices for the seven sample participants? Formulate and solve this 0 – 1 integer programming problem.

Example: Product Design & Market Share

- Decision Variables

  There are 7 l_{ij} decision variables, one for each level of attribute.

  > lij = 1 if Lucas chooses level i for attribute j;
  > 0 otherwise.

  There are 7 Y_k decision variables, one for each consumer in the sample.

  > Y_k = 1 if consumer k chooses the Lucas brand;
  > 0 otherwise

Example: Product Design & Market Share

- Objective Function

  Maximize the number of consumers preferring the Lucas brand desk.

$$\text{Max} \quad Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 + Y_7$$

Example: Product Design & Market Share

- Constraints

There is one constraint for each consumer in the sample.

$$
\begin{aligned}
5l_{11} + 26l_{21} + 20l_{31} + 18l_{12} + 11l_{22} + 17l_{13} + 10l_{23} - 50Y_1 &\geq 1\\
18l_{11} + 11l_{21} + 5l_{31} + 12l_{12} + 16l_{22} + 15l_{13} + 26l_{23} - 50Y_2 &\geq 1\\
4l_{11} + 16l_{21} + 22l_{31} + 7l_{12} + 13l_{22} + 11l_{13} + 19l_{23} - 50Y_3 &\geq 1\\
12l_{11} + 8l_{21} + 4l_{31} + 18l_{12} + 9l_{22} + 22l_{13} + 14l_{23} - 50Y_4 &\geq 1\\
19l_{11} + 9l_{21} + 3l_{31} + 4l_{12} + 14l_{22} + 30l_{13} + 19l_{23} - 50Y_5 &\geq 1\\
6l_{11} + 15l_{21} + 21l_{31} + 8l_{12} + 17l_{22} + 20l_{13} + 11l_{23} - 50Y_6 &\geq 1\\
9l_{11} + 6l_{21} + 3l_{31} + 13l_{12} + 5l_{22} + 16l_{13} + 28l_{23} - 50Y_7 &\geq 1
\end{aligned}
$$

Example: Product Design & Market Share

- Constraints

  There is one constraint for each attribute.

$$l_{11} + l_{21} + l_{31} = 1$$
$$l_{12} + l_{22} = 1$$
$$l_{13} + l_{23} = 1$$

Example: Product Design & Market Share

- Optimal Solution

Lucas should choose these product features:

1 file drawer ($l_{21} = 1$)
No pullout writing boards ($l_{22} = 1$)
Simulated wood finish ($l_{13} = 1$)

Three sample participants would choose the Lucas design:

Participant 1 ($Y_1 = 1$)
Participant 5 ($Y_5 = 1$)
Participant 6 ($Y_6 = 1$)

Modeling Flexibility Provided by 0-1 Variables

- When x_i and x_j represent binary variables designating whether projects i and j have been completed, the following special constraints may be formulated:
  - At most <u>*k* out of *n*</u> projects will be completed:
    $$\sum_j x_j \leq k$$
  - Project j is <u>conditional</u> on project i:
    $$x_j - x_i \leq 0$$
  - Project i is a <u>corequisite</u> for project j:
    $$x_j - x_i = 0$$
  - Projects i and j are <u>mutually exclusive</u>:
    $$x_i + x_j \leq 1$$

Example: Metropolitan Microwaves

Metropolitan Microwaves, Inc. is planning to expand its sales operation by offering other electronic appliances. The company has identified seven new product lines it can carry. Relevant information about each line follows on the next slide.

Example: Metropolitan Microwaves

Product Line	Initial Invest.	Floor Space (Sq.Ft.)	Exp. Rate of Return
1. TV/DVRs	$ 6,000	125	8.1%
2. TVs	12,000	150	9.0
3. Projection TVs	20,000	200	11.0
4. DVRs	14,000	40	10.2
5. DVD Players	15,000	40	10.5
6. Video Games	2,000	20	14.1
7. Desktop Computers	32,000	100	13.2

Example: Metropolitan Microwaves

Metropolitan has decided that they should not stock projection TVs unless they stock either TV/DVRs or TVs. Also, they will not stock both DVRs and DVD players, and they will stock video games if they stock TVs. Finally, the company wishes to introduce at least three new product lines.

If the company has $45,000 to invest and 420 sq. ft. of floor space available, formulate an integer linear program for Metropolitan to maximize its overall expected return.

Example: Metropolitan Microwaves

- Define the Decision Variables

  x_j = 1 if product line j is introduced;

  = 0 otherwise.

  where:

  Product line 1 = TV/DVRs
  Product line 2 = TVs
  Product line 3 = Projection TVs
  Product line 4 = DVRs
  Product line 5 = DVD Players
  Product line 6 = Video Games
  Product line 7 = Desktop Computers

Example: Metropolitan Microwaves

- Define the Decision Variables

  x_j = 1 if product line j is introduced;
  = 0 otherwise.

- Define the Objective Function

  Maximize total expected return:

  $$\text{Max} \quad .081(6000)x_1 + .09(12000)x_2 + .11(20000)x_3 + .102(14000)x_4 + .105(15000)x_5 + .141(2000)x_6 + .132(32000)x_7$$

Example: Metropolitan Microwaves

- Define the Constraints

1) Money:

$$6x_1 + 12x_2 + 20x_3 + 14x_4 + 15x_5 + 2x_6 + 32x_7 \leq 45$$

2) Space:

$$125x_1 + 150x_2 + 200x_3 + 40x_4 + 40x_5 + 20x_6 + 100x_7 \leq 420$$

3) Stock projection TVs only if stock TV/DVRs or TVs:

$$x_1 + x_2 > x_3 \quad \text{or} \quad x_1 + x_2 - x_3 \geq 0$$

Example: Metropolitan Microwaves

- Define the Constraints (continued)

  4) Do not stock both DVRs and DVD players:

  $$x_4 + x_5 \leq 1$$

  5) Stock video games if they stock TV's:

  $$x_2 - x_6 \geq 0$$

  6) Introduce at least 3 new lines:

  $$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \geq 3$$

  7) Variables are 0 or 1:

  $$x_j = 0 \text{ or } 1 \text{ for } j = 1, , , 7$$

Example: Metropolitan Microwaves

- Optimal Solution

Introduce:
TV/DVRs, Projection TVs, DVD Players

Do Not Introduce:
TVs, DVRs, Video Games, Desktop Computers

Total Expected Return:
$4,261

Cautionary Note About Sensitivity Analysis

- Sensitivity analysis often is more crucial for ILP problems than for LP problems.
- A small change in a constraint coefficient can cause a relatively large change in the optimal solution.
- Recommendation: Resolve the ILP problem several times with slight variations in the coefficients before choosing the "best" solution for implementation.