Linear Programming III: Duality and Sensitivity Analysis

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Duality

- Every LPP has a *dual*, the given LPP is *primal*
- If primal is a max problem the dual is a min problem
- An *m*-variable *n*-constraint primal has an *n*-variable *m*-constraint dual
- The dual is obtained by transposing the three matrices involved in the primal problem

Duality

(...continued)

- Before writing dual for an LPP, make sure
  1. All variables are non-negative
     - Replace every unrestricted variable by difference of two non-negative variables
  2. All constraints are ≤ type for a max problem/ ≥ type for a min problem
     - If a constraint is in a direction opposite to the one desired, then multiply it both sides by -1 and reverse the direction of inequality
     - Replace a constraint involving "=" sign with a pair of constraints with identical LHS and RHS values: one with ≥ sign and the other with ≥ sign

Primal-Dual Relationship

(Assuming a max primal problem)

Primal	Dual
Max	Min
No. of variables	No. of constraints
No. of constraints	No. of variables
≤ type constraint	Non-negative variable
= type constraint	Unrestricted variable
Unrestricted variable	= type constraint
Objective function value for *j*th variable	RHS constant for *j*th constraint
RHS constant for *i*th constraint	Objective function value for *i*th variable
Co-efficient (a_{ij}) for *j*th variable in the *i*th constraint	Co-efficient (a_{ij}) for *i*th variable in the *j*th constraint

Writing the Dual: Example 1

- **Primal**

  Maximize $Z = 22x_1 + 30x_2 + 32x_3$

  Subject to $5x_1 + 3x_2 + 8x_3 \leq 70$

  $2x_1 - 8x_2 + 9x_3 \leq 86$

  $x_1, x_2, x_3 \geq 0$

  Here both conditions are satisfied.

- **Dual**

  Minimize G = 70y1 + 86y2

  Subject to 5y1 + 2y2 ≥ 22

  3y1 – 8y2 ≥ 30

  8y1 + 9y2 ≥ 32

  y1, y2 ≥ 0

  - Here x1 , x2 , and x3 are **primal variables** and y1 and y2 are **dual variables**
  - The primal has 3 variables and 2 constraints, while the dual has 2 variables and 3 constraints

Writing the Dual: Example 2

- Primal

Maximize $Z = 3x_1 + 5x_2 + 7x_3$

Subject to $2x_1 + 4x_2 + 3x_3 \leq 40$

$-4x_1 + 5x_2 - 3x_3 \geq 25$

$x_1 + 2x_2 + 5x_3 = 15$

$x_1, x_2 \geq 0$, x_3: unrestricted in sign

- To write the dual, first multiply second constraint by -1; replace the third constraint by a pair of inequalities in opposite directions; and replace the unrestricted variable x3 = x4 – x5

The problem becomes:

Maximize Z = 3x1 + 5x2 + 7x4 – 7x5

Subject to 2x1 + 4x2 + 3x4 – 3x5 ≤ 40

4x1 – 5x2 –3 x4 + 3x5 ≤ – 25

x1 + 2x2 + 5x4 – 5x5 ≤ 15

– x1 – 2x2 – 5x4 + 5x5 ≤ – 15

x1, x2, x4, x5 ≥ 0

Example 2 *(...continued)*

With dual variables y_1, y_2, y_3 and y_4,

Minimize $G = 40y_1 - 25y_2 + 15y_3 - 15y_4$

Subject to $2y_1 + 4y_2 + y_3 - y_4 \geq 3$

$4y_1 - 5y_2 + 2y_3 - 2y_4 \geq 5$

$3y_1 - 3y_2 + 5y_3 - 5y_4 \geq 7$

$-3y_1 + 3y_2 - 5y_3 + 5y_4 \geq -7$

y_1, y_2, y_3 and $y_4 \geq 0$

- Now, let $y_3 - y_4 = y_5$ and combining the last two constraints and re-writing them instead as an equation, the **Dual** is:

  Minimize $G = 40y_1 - 25y_2 + 15y_5$

  Subject to $2y_1 + 4y_2 + y_5 \geq 3$

  $4y_1 - 5y_2 + 2y_5 \geq 5$

  $3y_1 - 3y_2 + 5y_5 = 7$

  $y_1, y_2 \geq 0$ and y_5 unrestricted in sign

- For an unrestricted variable in the primal, a constraint in dual involves an equation, while for an equation in the primal, there is an unrestricted variable

Primal-Dual Relationship

- If feasible solutions exist for both primal and dual, the objective function values of their optimal solutions are equal

- The solution to the dual can be read from the Δ_{ij} values of the slack/surplus variables in the optimal solution tableau of the primal

- If primal has unbounded solution, the dual has no feasible solution

Primal and Dual Solutions

- Example 3.2 Data
  - Primal Problem

$$\begin{aligned} \text{Max} \quad & Z = 5x_1 + 10x_2 + 8x_3 && \text{Contribution} \\ \text{St} \quad & 3x_1 + 10x_2 + 2x_3 \leq 60 && \text{Fabrication Hrs} \\ & 4x_1 + 4x_2 + 4x_3 \leq 72 && \text{Finishing Hrs} \\ & 2x_1 + 4x_2 + 4x_3 \leq 100 && \text{Packaging Hrs} \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

  - Dual Problem

$$\begin{aligned} \text{Min } G = {} & 60y_1 + 72y_2 + 100y_3 \\ \text{St} \quad & 3y_1 + 4y_2 + 2y_3 \geq 5 \\ & 10y_1 + 4y_2 + 4y_3 \geq 10 \\ & 2y_1 + 4y_2 + 4y_3 \geq 8 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

Primal and Dual Solutions

Simplex Tableau: Optimal Solution

Basis	x_1	x_2	x_3	S_1	S_2	S_3	b_i
x_2 10	1/3	1	0	1/3	-1/6	0	8
x_3 8	2/3	0	1	-1/3	5/12	0	10
S_3 0	-8/3	0	0	1/3	-17/12	1	18
c_j	5	10	8	0	0	0	
Sol	0	8	10	0	0	18	160
Δ_j	-11/3	0	0	-2/3	-5/3	0	

- Solution to primal problem:

  $x_1 = 0$, $x_2 = 8$ and $x_3 = 10$

  $Z = 5\times0 + 10\times8 + 8\times10 = 160$

- Solution to dual problem:

  $y_1 = 2/3$, $y_2 = 5/3$ and $y_3 = 0$

  $G = 60\times2/3 + 72\times5/3 + 100\times0 = 160$

Economic Interpretation of Dual

- The Δ_j values corresponding to slack/surplus variables in the optimal solution indicate marginal profitability (or shadow prices) of the resources they represent

- A resource unused fully has zero shadow price

- A unit change in the availability of a resource changes the objective function value by an amount of its shadow price

Sensitivity Analysis

- RHS Ranging
  - Shadow prices are valid only over certain ranges
  - The range for each resource is determined by b_i and a_{ij} values in the optimal solution tableau

- Changes in objective function co-efficients
  - Within certain limits, such changes do not induce changes in the optimal mix
  - The limits are determined by Δ_j and the appropriate a_{ij} values in the optimal solution tableau

Economic Interpretation

Simplex Tableau: Optimal Solution

Basis	x_1	x_2	x_3	S_1	S_2	S_3	b_i
x_2 10	1/3	1	0	1/3	-1/6	0	8
x_3 8	2/3	0	1	-1/3	5/12	0	10
S_3 0	-8/3	0	0	1/3	-17/12	1	18
C_j	5	10	8	0	0	0	
Sol	0	8	10	0	0	18	160
Δ_j	-11/3	0	0	-2/3	-5/3	0	

- Each hour in Fabrication is worth Rs 2/3 so that an increase of one hour would increase profit by Rs 2/3 and a decrease of one hour would result in a decrease of Rs 2/3
- Similarly, each hour in Finishing is worth Rs 5/3
- The marginal profitability of Packaging hours is 0 since there is unutilized capacity of 18 hours. Further addition to capacity will not add to profit and reducing capacity will also not affect it

RHS Ranging

- Fabrication Hours

b_i	a_{ij}	b_i/a_{ij}
8	1/3	24
10	-1/3	-30
18	1/3	54

← Least positive

← Least negative

  - Range: 60-24 = 36 to 60-(-30)=90
  - Shadow price of Rs. 2/3 is valid over the range 36 to 90 hours

- Finishing Hours

b_i	a_{ij}	b_i/a_{ij}
8	-1/6	-48
10	5/12	24
18	-17/12	-12.71

← Least positive

← Least negative

  - Range: 72-24 to 72+12.71 OR 48 to 84.71

- Packaging Hours
  - 100-18=72
  - Therefore, range is 72 to ∞

Changes in Objective Function Co-efficients

For x_2 (a basic variable)						
Δ_j	-11/3	0	0	-2/3	-5/3	0
a_{ij}	1/3	1	0	1/3	-1/6	0
Ratio	-11	0	-	-2	10	-

Least negative (-2) — Least positive (10)

Range: 10-2 to 10+10 OR Rs. 8 to Rs. 20

For x_3 (a basic variable)						
Δ_j	-11/3	0	0	-2/3	-5/3	0
a_{ij}	2/3	0	1	-1/3	5/12	0
Ratio	-11/2	-	0	2	-4	-

Least positive (2) — Least negative (-4)

Range: 8-4 to 8+2 OR Rs. 4 to Rs. 10

For x_1 (a non-basic variable)

Range: $-\infty$ to 5+11/3 OR Rs $-\infty$ to Rs 8.67

Multiple Choice Questions

- Mark the wrong statement:
  1. If the primal is a minimisation problem, its dual will be a maximisation problem.
  2. Columns of the constraint co-efficients in the primal problem become columns of the constraint co-efficients in the dual.
  3. For an unrestricted primal variable, the associated dual constraint is an equation.
  4. If a constraint in a maximisation type of primal problem is a “less-than-or-equal-to” type, the corresponding dual variable is non-negative.

Multiple Choice Questions

- Mark the wrong statement:
  1. The dual of the dual is primal.
  2. An equation in a constraint of a primal problem implies the associated variable in the dual problem to be unrestricted.
  3. If a primal variable is non-negative, the corresponding dual constraint is an equation.
  4. The objective function co-efficients in the primal problem become right-hand sides of constraints of the dual.

Multiple Choice Questions

- Choose the wrong statement:
  1. In order that dual to an LPP may be written, it is necessary that it has at least as many constraints as the number of variables.
  2. The dual represents an alternate formulation of LPP with decision variables being implicit values.
  3. The optimal values of the dual variables can be obtained by inspecting the optimal tableau of the primal problem as well.
  4. Sensitivity analysis is carried out having reference to the optimal tableau alone.

Multiple Choice Questions

- Choose the incorrect statement:
  1. All scarce resources have marginal profitability equal to zero.
  2. Shadow prices are also known as imputed values of the resources.
  3. A constraint $3x_1 - 7x_2 + 13x_3 - 4x_4 \geq -10$ can be equivalently written as $-3x_1 + 7x_2 - 13x_3 + 4x_4 \leq 10$.
  4. If all constraints of a minimisation problem are $\geq$ type, then all dual variables are non-negative.

Multiple Choice Questions

- To write the dual, it should be ensured that
  i. All the primal variables are non-negative
  ii. All the bi values are non-negative
  iii. All the constraints are $\leq$ type if it is maximisation problem and $\geq$ type if it is a minimisation problem

1. i and ii
2. ii and iii
3. i and iii
4. i, ii and iii

Multiple Choice Questions

- Mark the wrong statement:

1. If the optimal solution to an LPP exists then the objective function values for the primal and the dual shall both be equal.
2. The optimal values of the dual variables are obtained from Δ_j values from slack/surplus variables, in the optimal solution tableau.
3. An *n*-variable *m*-constraint primal problem has an *m*-variable *n*-constraint dual.
4. If a constraint in the primal problem has a negative b_i value, its dual cannot be written.

Multiple Choice Questions

- Mark the wrong statement:

1. The primal and dual have equal number of variables.
2. The shadow price indicates the change in the value of the objective function, per unit increase in the value of the RHS.
3. The shadow price of a non-binding constraint is always equal to zero.
4. The information about shadow price of a constraint is important since it may be possible to purchase or otherwise acquire additional units of the concerned resource.

Multiple Choice Questions

- Choose the most correct of the following statements relating to primal-dual linear programming problems:

1. Shadow prices of resources in optimal solution of the primal are optimal values of the dual variables.
2. The optimal values of the objective functions of primal and dual are the same.
3. If the primal problem has unbounded solution, the dual problem would have infeasibility.
4. All of the above.

Multiple Choice Questions

- In linear programming context, the sensitivity analysis is a technique to

1. allocate resources optimally.
2. minimise cost of operations.
3. spell out relation between primal and dual.
4. determine how optimal solution to LPP changes in response to problem inputs.

Multiple Choice Questions

- Which of these does not hold for the 100% Rule?
  1. It can be applied when simultaneous changes in the objective function and the right-hand sides of the constraints are to be jointly considered.
  2. For all objective function co-efficients changed, if the sum of percentages of allowable increases and decreases does not exceed 100 per cent, then the optimal solution will not change.
  3. For all RHS values changed, if the sum of percentages of allowable increases and decreases does not exceed 100 per cent, then the shadow prices will not change.
  4. It is applied for considering simultaneous changes in the objective function co-efficients or simultaneous changes in the RHS values.