# OPERATIONS RESEARCH

## Linear Programming – II Graphical Method

Topic 2

ITM BUSINESS SCHOOL
OPERATIONS AND SUPPLY CHAIN MANAGEMENT
SEMESTER II
VIJAYANTA PAWASE

# Linear Programming - II

## *Graphical Method*

# Introduction

- There are two methods available to find an optimal solution to a Linear Programming Problem.
- One is a graphical method and the other is a simplex method.
- **Graphical method can be used only for a two variables problem** i.e., a problem that involves two decision variables.
- The two axes of the graph (X & Y axis) represent the two decision variables $X_1$ & $X_2$.

# Methodology of Graphical Method

**Step 1:**

- Formulation of LPP (Linear Programming Problem)

**Step 2:**

- Determination of each axis.

**Step 3:**

- Finding coordinates of constraint lines to represent constraint lines on the graph.

# Methodology of Graphical Method

## Step 4:

- Representing constraint lines on the graph

## Step 5:

- Identification of Feasible Region

## Step 6:

- Finding the optimal solution

# Methodology of Graphical Method

- **Step 1:** Formulation of LPP (Linear Programming Problem)
- **Step 2:** Determination of each axis.
- **Step 3:** Finding coordinates of constraint lines to represent constraint lines on the graph.
- **Step 4:** Representing constraint lines on the graph
- **Step 5:** Identification of Feasible Region
- **Step 6:** Finding the optimal solution

# Methodology of Graphical Method

## Step 1: Formulation of LPP (Linear Programming Problem)

- Use the given data to formulate the LPP.

**Maximization:**

**Example 1:** A company manufactures two products A and B. Both products are processed on two machines $M_1$ & $M_2$.

| | $M_1$ | $M_2$ |
|---|---|---|
| A | 6 Hrs./unit | 2 Hrs/unit |
| B | 4 Hrs./unit | 4 Hrs./unit |
| Availability | 7200 Hrs./month | 4000 Hrs./month |

Profit per unit for A is Rs. 100 and for B is Rs. 80. Find out monthly production of A and B to maximize profit by graphical method.

# Methodology of Graphical Method

## Maximization:

**Example 1:** A company manufactures two products A and B. Both products are processed on two machines $M_1$ & $M_2$.

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Profit per unit for A is Rs. 100 and for B is Rs. 80. Find out monthly production of A and B to maximize profit by graphical method.

**Step 1: Formulation as LPP**

$X_1$ = No. of units of A/month

$X_2$ = No. of units of B/month

**Max. Z = $100X_1 + 80X_2$**

**Subject to constraints:**

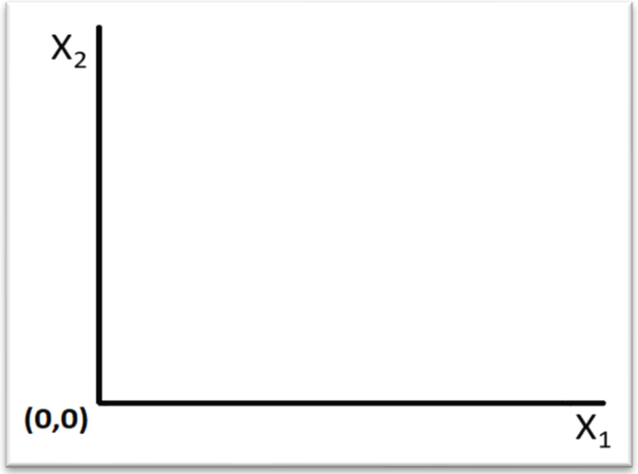
$$6X_1 + 4X_2 \leq 7200$$

$$2X_1 + 4X_2 \leq 4000$$

$$X_1, X_2 \geq 0$$

**Step 2: Determination of each axis.**

- Horizontal (X) axis: Product A ($X_1$)
- Vertical (Y) axis: Product B ($X_2$)



# Methodology of Graphical Method

Step 3: Finding coordinates of constraint lines to represent constraint lines on the graph.

- The constraints are present in the form of inequality (≤). We should convert them into equality to obtain coordinates.

**(1) Constraint No.1: $6X_1 + 4X_2 \leq 7200$**

- Converting into equality:

$6X_1 + 4X_2 = 7200$

$X_1$ is the intercept on X axis and $X_2$ is the intercept on Y axis.

- To find $X_1$, let $X_2 = 0$

$6X_1 = 7200$ Therefore, $X_1$, = 1200, $X_2 = 0$

- To find $X_2$, let $X_1 = 0$

$4X_2 = 7200$ Therefore, $X_2 = 1800$, $X_1 = 0$

Hence the two points which make the constraint line are: **(1200,0) and (0,1800)**

# Methodology of Graphical Method

Step 3: Finding coordinates of constraint lines to represent constraint lines on the graph.

- The constraints are present in the form of inequality (≤). We should convert them into equality to obtain coordinates.

**(2) Constraint No.2: $2X_l + 4X_2 \leq 4000$**

- To find $X_1$, let $X_2 = 0$

$2X_l = 4000$ Therefore, $X_1 = 2000$, $X_2 = 0$, (2000,0)

- To find $X_2$, let $X_1 = 0$

$4X_2 = 4000$ Therefore, $X_2 = 1000$, $X_1 = 0$, (0,1000)

Each constraint will be represented by a single straight line on the graph. There are two constraints, hence there will be two straight lines.

The co-ordinates of points are:

(1) Constraint No.1: (1200,0) and (0,1800)

(2) Constraint No.2: (2000,0) and (0,1000)

# Methodology of Graphical Method

## Step 4: Representing constraint lines on the graph

- To mark the points on the graph, we need to select the appropriate scale.

Which scale to take will depend on maximum value of $X_1$ & $X_2$ from co-ordinates.

- For $X_1$, we have 2 values → 1200 and 2000.

Therefore, Max. value for $X_2$ = 2000

- For $X_2$, we have 2 values → 1800 and 1000

Therefore, Max. value for $X_2$ = 1800

Assuming that we have a graph paper 20 x 30 cm. We need to accommodate our lines such that for X axis, maximum value of 2000 contains in 20 cm.

Scale: 1 cm = 200 units

2000 units = 10 cm (X axis)

1800 units = 9cm (Y axis)

The scale should be such that the diagram should not be too small.

# Methodology of Graphical Method

## Step 4: Representing constraint lines on the graph

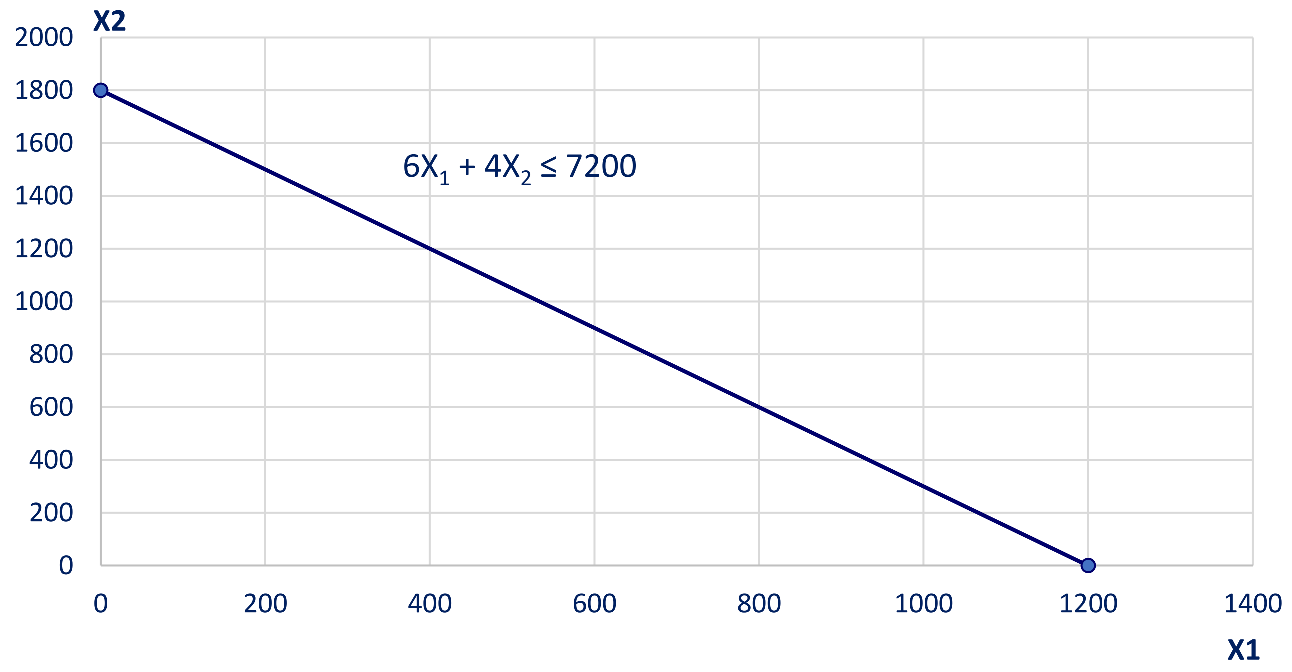
- To mark the points on the graph, we need to select the appropriate scale.

Constraint No.1:

The line joining the two points (1200,0) and (0,1800) represents the constraint $6X_1 + 4X_2 \leq 7200$.

Every point on the line will satisfy the equation (equality) $6X_1 + 4X_2 = 7200$.

Every point below the line will satisfy the inequality (less than) $6X_1 + 4X_2 < 7200$.



# Methodology of Graphical Method

## Step 4: Representing constraint lines on the graph

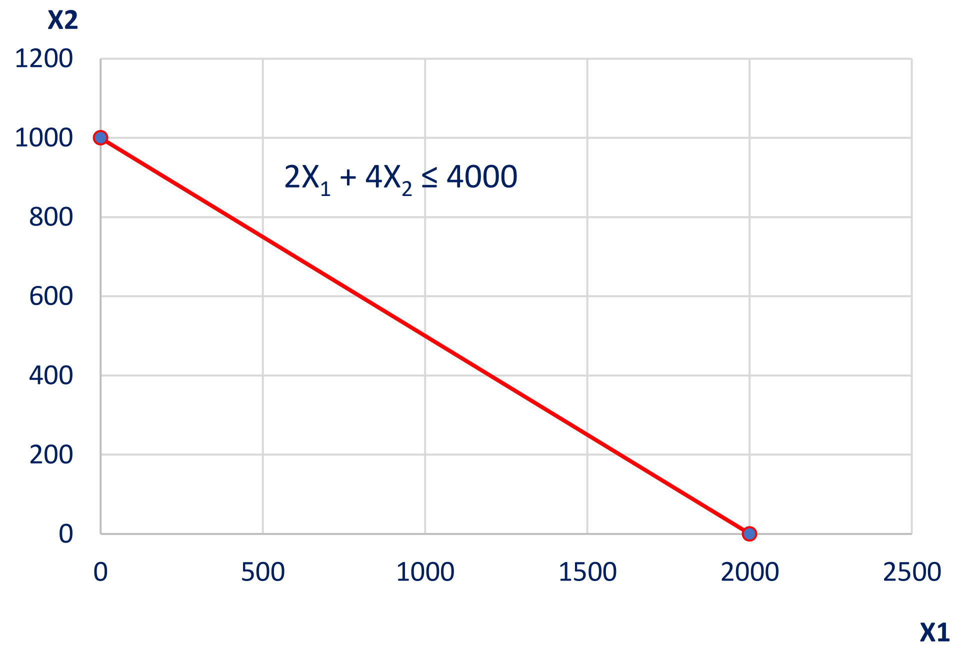
- To mark the points on the graph, we need to select the appropriate scale.

Constraint No.2:

The line joining the two points (2000,0) and (0,1000) represents the constraint $2X_1 + 4X_2 \leq 4000$.

Every point on the line will satisfy the equation (equality) $2X_1 + 4X_2 = 4000$.

Every point below the line will satisfy the inequality (less than) $2X_1 + 4X_2 < 4000$.

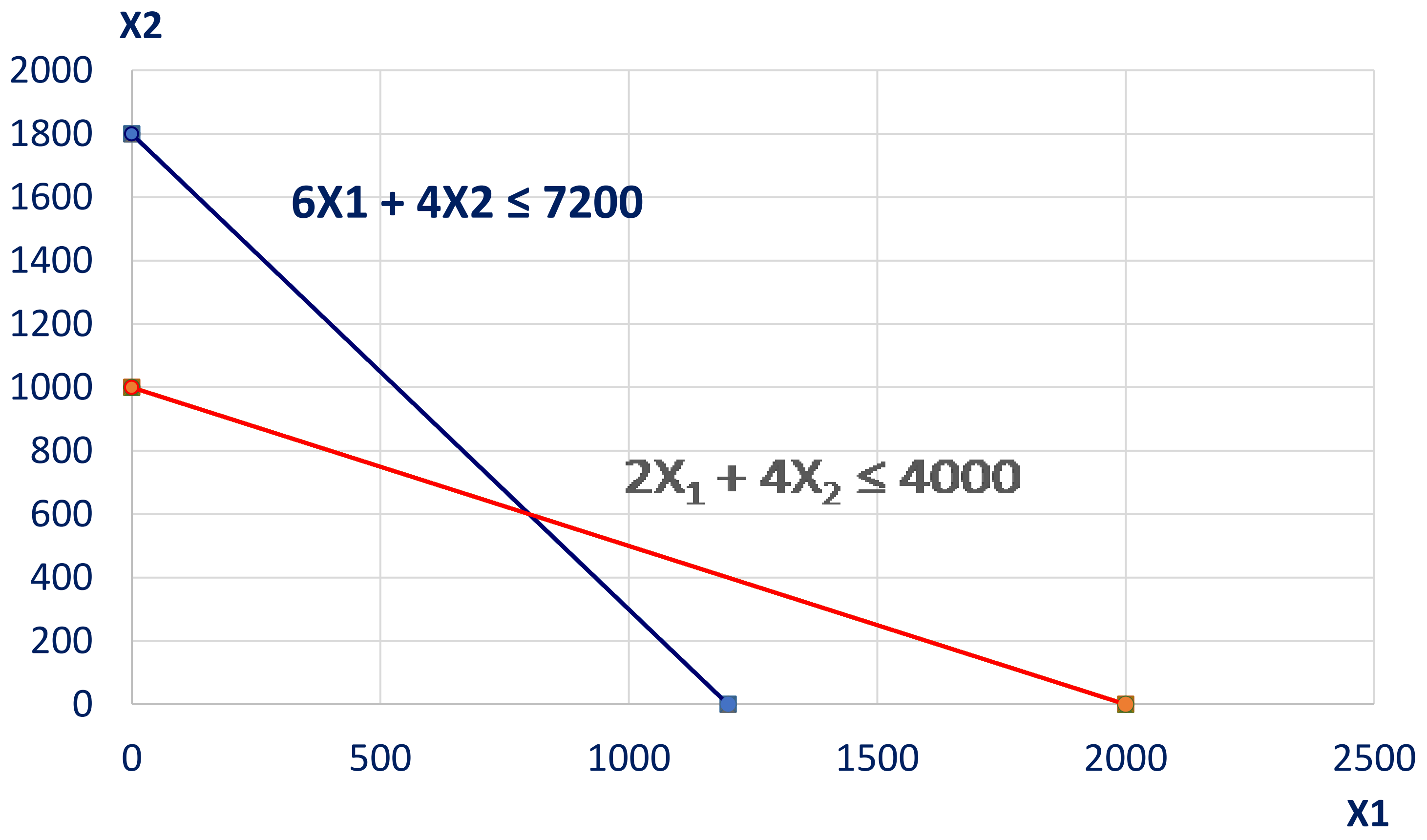


# Methodology of Graphical Method

**Step 4: Representing constraint lines on the graph**

- To mark the points on the graph, we need to select the appropriate scale.

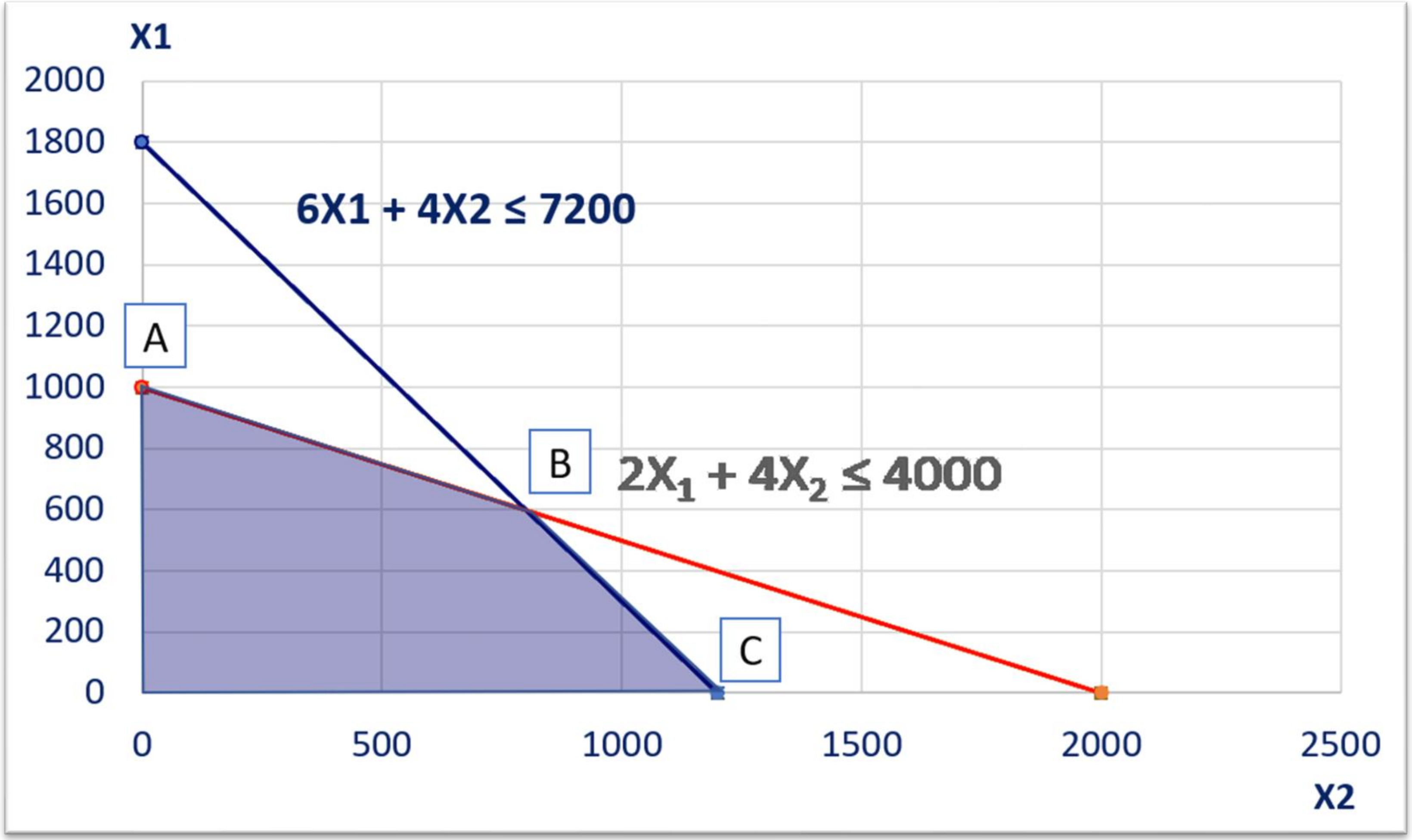
Now the graph will look like this:



# Methodology of Graphical Method

## Step 5: Identification of Feasible Region

- The feasible region is the region bounded by constraint lines. All points inside the feasible region or on the boundary of the feasible region or at the comers of the feasible region satisfy all constraints.




Both the constraints are 'less than or equal to' (≤) type. Hence, the feasible region should be **inside both constraint lines.**

Hence, the feasible region is the polygon OABC. '0' is the origin whose co-ordinates are (0,0). O, A, B and C are called vertices of the feasible region.

# Methodology of Graphical Method

**What is meant by the feasible region of a solution in the LPP graphical method?**

- The feasible region of the solution means **that part of the graphical solution satisfies all the constraints of the problem.**
- There may be different types of constraints in the problem e.g., ≥, ≤, etc.
- The feasible region is the common region to all constraints.
- To find the feasible region, **we should eliminate all those parts of the solution which are not common to all constraints.** The remaining region is the one that is the feasible region of solution.

| Type of constraint | Location of feasible region |
|---|---|
| ≤ | Inside the constraint line |
| ≥ | Outside the constraint line |
| = | On the constraint line |

# Methodology of Graphical Method

## Step 6: Finding the optimal solution

- The optimal 'solution always lies at one of the vertices or corners of the feasible region.
- To find optimal solution:
- We use corner point method. We find co-ordinates (X1 and X2 values) for each vertex or corner point. From this we find 'Z' value for each corner point.

| Vertex | Co-ordinates | $Z = 100X_1 + 80X_2$ |
|---|---|---|
| 0 | $X_1 = 0; X_2 = 0$ | Z =0 |
| | From Graph | |
| A | $X_1 = 0, X_2 = 1000$ | Z = Rs. 80,000 |
| | From Graph | |
| B | $X_1 = 800, X_2 = 600$ | Z = Rs.128,000 |
| | From simultaneous equations | |
| C | $X_1 = 1200, X_2 = 0$ | Z = Rs. 120,000 |
| | From Graph | |

For B → B is at the intersection of two constraint lines $6X_l + 4X_2 = 7200$ and 2X1 + 4X2 = 4000. Hence, values of X1 and $X_2$ at B must satisfy both the equations.

**Max. Z = Rs. 128,000 (At point B)**

**Solution:**
Optimal profit = Max. Z = Rs. 128,000
**Product Mix:**
$X_1$ = No. of units of A/month = 800
$X_2$ = No. of units of B/month = 600

# Methodology of Graphical Method

## Step 1: Formulation of LPP (Linear Programming Problem)

- Use the given data to formulate the LPP.

## Minimization:

**Example 2:** A firm is engaged in animal breeding. The animals are to be given nutrition supplements every day. There are two products A and B which contain the three required nutrients.

| Nutrients | Quantity/unit | | Minimum requirement |
|---|---|---|---|
| | A | B | |
| 1 | 72 | 12 | 216 |
| 2 | 6 | 24 | 72 |
| 3 | 40 | 20 | 200 |

Product cost per unit is - A: Rs. 40, B: Rs. 80. Find out the quantity of product A & B to be given to provide minimum nutritional requirement.

# Methodology of Graphical Method

## Minimization:

**Example 2:** A firm is engaged in animal breeding. The animals are to be given nutrition supplements every day. There are two products A and B which contain the three required nutrients.

| Nutrients | Quantity/unit | | Minimum requirement |
|---|---|---|---|
| | A | B | |
| 1 | 72 | 12 | 216 |
| 2 | 6 | 24 | 72 |
| 3 | 40 | 20 | 200 |

Product cost per unit is - A: Rs. 40, B: Rs. 80. Find out the quantity of product A & B to be given to provide minimum nutritional requirement.

**Step 1: Formulation as LPP**

$X_1$ = No. of units of A

$X_2$ = No. of units of B

**Z = Total cost**

**Min. $Z = 40X_1 + 80X_2$**

**Subject to constraints:**

$72X_1 + 12X_2 \geq 216$

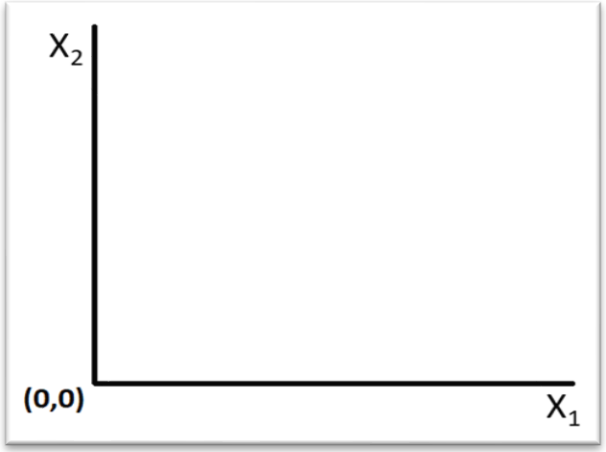
$6X_1 + 24X_2 \geq 72$

$40X_1 + 20X_2 \geq 200$

$X_1, X_2 \geq 0$

**Step 2: Determination of each axis.**

- Horizontal (X) axis: Product A ($X_1$)
- Vertical (Y) axis: Product B ($X_2$)



# Methodology of Graphical Method

Step 3: Finding coordinates of constraint lines to represent constraint lines on the graph.

- All constraints are 'greater than or equal to' type. We should convert them into equality.

**(1) Constraint No.1: $72X_1 + 12X_2 \geq 216$**

- Converting into equality:

$72X_1 + 12X_2 = 216$

$X_1$ is the intercept on X axis and $X_2$ is the intercept on Y axis.

- To find $X_1$, let $X_2 = 0$

$72X_1 = 216$ Therefore, $X_1, = 3, X_2 = 0$

- To find $X_2$, let $X_1 = 0$

$12X_2 = 216$ Therefore, $X_2 = 18, X_1 = 0$

Hence the two points which make the constraint line are: **(3,0) and (0,18)**

# Methodology of Graphical Method

Step 3: Finding coordinates of constraint lines to represent constraint lines on the graph.

- All constraints are 'greater than or equal to' type. We should convert them into equality.

**(2) Constraint No.2: $6X_1 + 24X_2 \geq 72$**

- Converting into equality:

$6X_1 + 24X_2 = 72$

$X_1$ is the intercept on X axis and $X_2$ is the intercept on Y axis.

- To find $X_1$, let $X_2 = 0$

$6X_1 = 72$ Therefore, $X_1$, $= 12$, $X_2 = 0$

- To find $X_2$, let $X_1 = 0$

$24X_2 = 72$ Therefore, $X_2 = 3$, $X_1 = 0$

Hence the two points which make the constraint line are: **(12,0) and (0,3)**

# Methodology of Graphical Method

Step 3: Finding coordinates of constraint lines to represent constraint lines on the graph.

- All constraints are 'greater than or equal to' type. We should convert them into equality.

**(3) Constraint No.3: $40X_1 + 20X_2 \geq 200$**

- Converting into equality:

$40X_1 + 20X_2 = 200$

$X_1$ is the intercept on X axis and $X_2$ is the intercept on Y axis.

- To find $X_1$, let $X_2 = 0$

$40X_1 = 200$ Therefore, $X_1$, $= 5$, $X_2 = 0$

- To find $X_2$, let $X_1 = 0$

$20X_2 = 200$ Therefore, $X_2 = 10$, $X_1 = 0$

Hence the two points which make the constraint line are: **(5,0) and (0,10)**

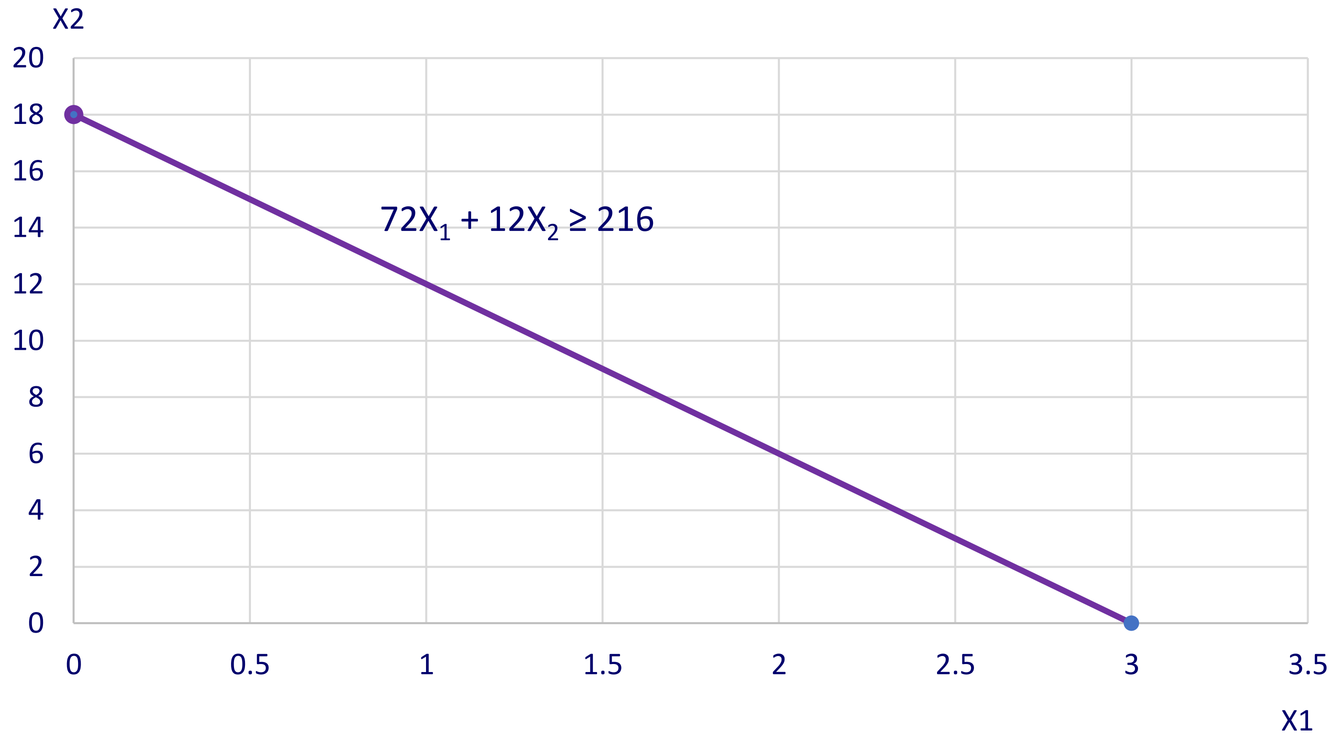
# Methodology of Graphical Method

## Step 4: Representing constraint lines on the graph

- To mark the points on the graph, we need to select the appropriate scale.

To select scale:
Maximum value for $X_1$ = 12
Maximum value for $X_2$ = 18
Scale: 1 cm = 1 unit

**Constraint No.1:**




Every point **on the line** will satisfy the equation (equality) $72X_1 + 12X_2 = 216$.

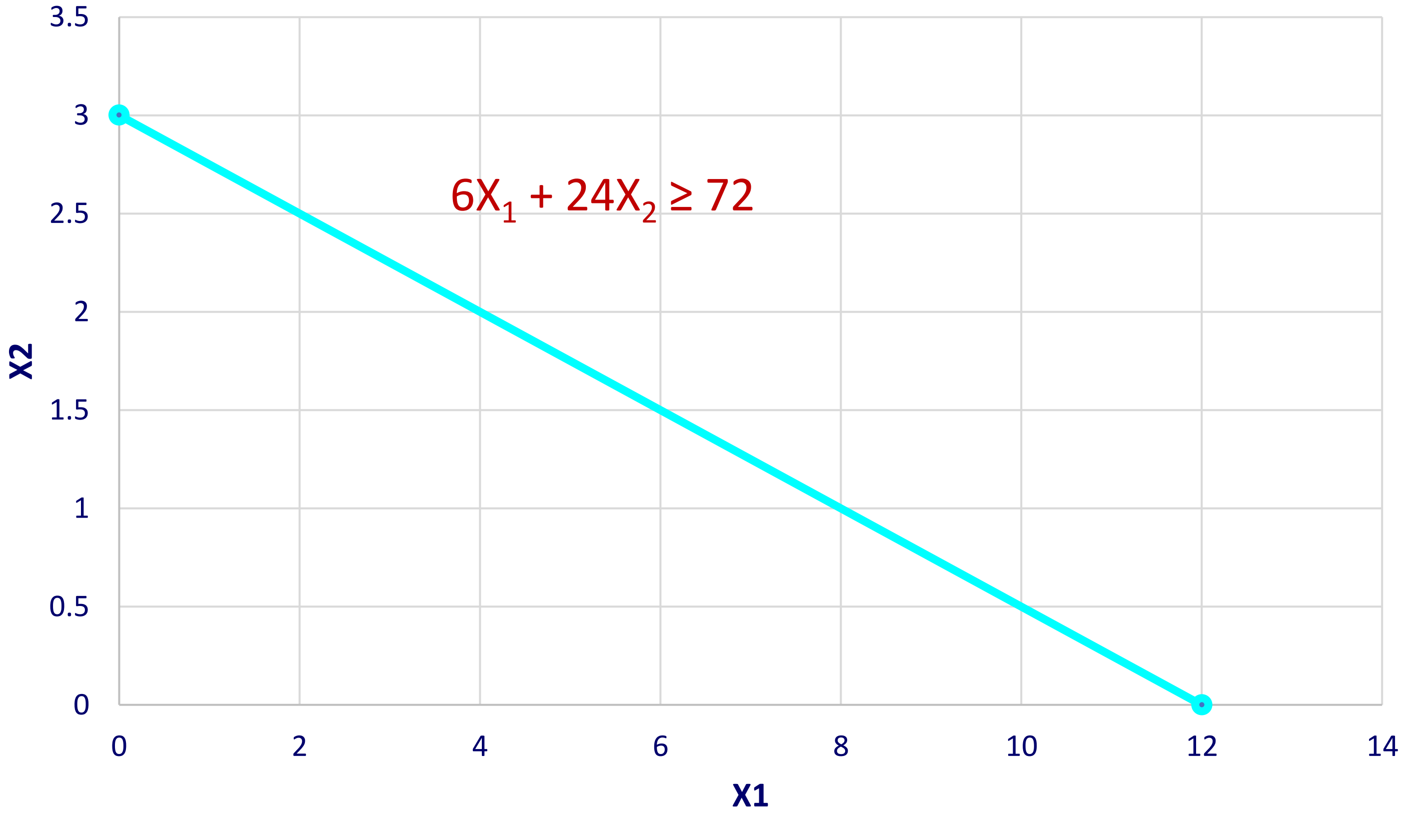
Every point **above the line** will satisfy the inequality (greater than) $72X_1 + 12X_2 > 216$.

# Methodology of Graphical Method

## Step 4: Representing constraint lines on the graph

- To mark the points on the graph, we need to select the appropriate scale.

**Constraint No.2:**

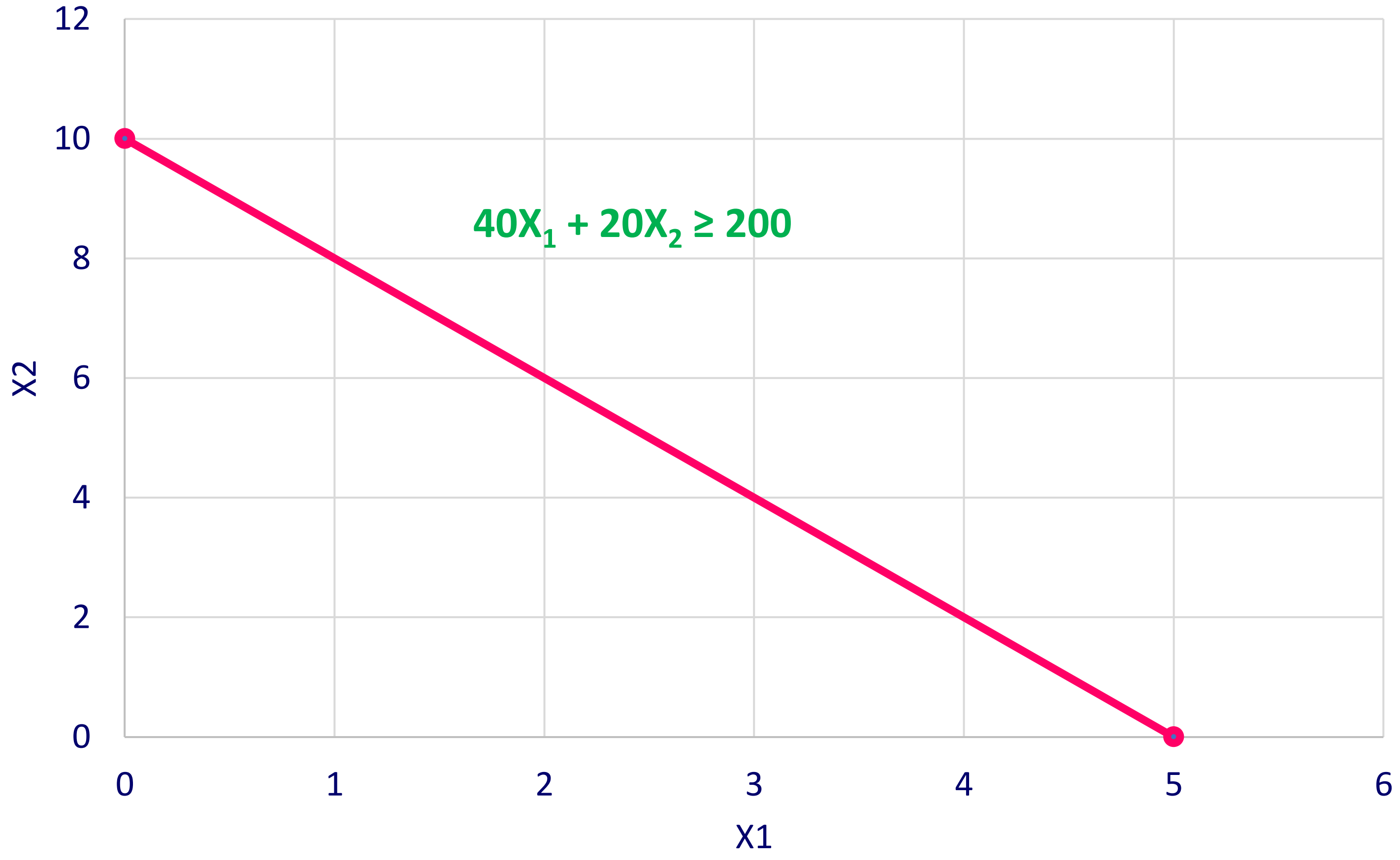


# Methodology of Graphical Method

## Step 4: Representing constraint lines on the graph

- To mark the points on the graph, we need to select the appropriate scale.

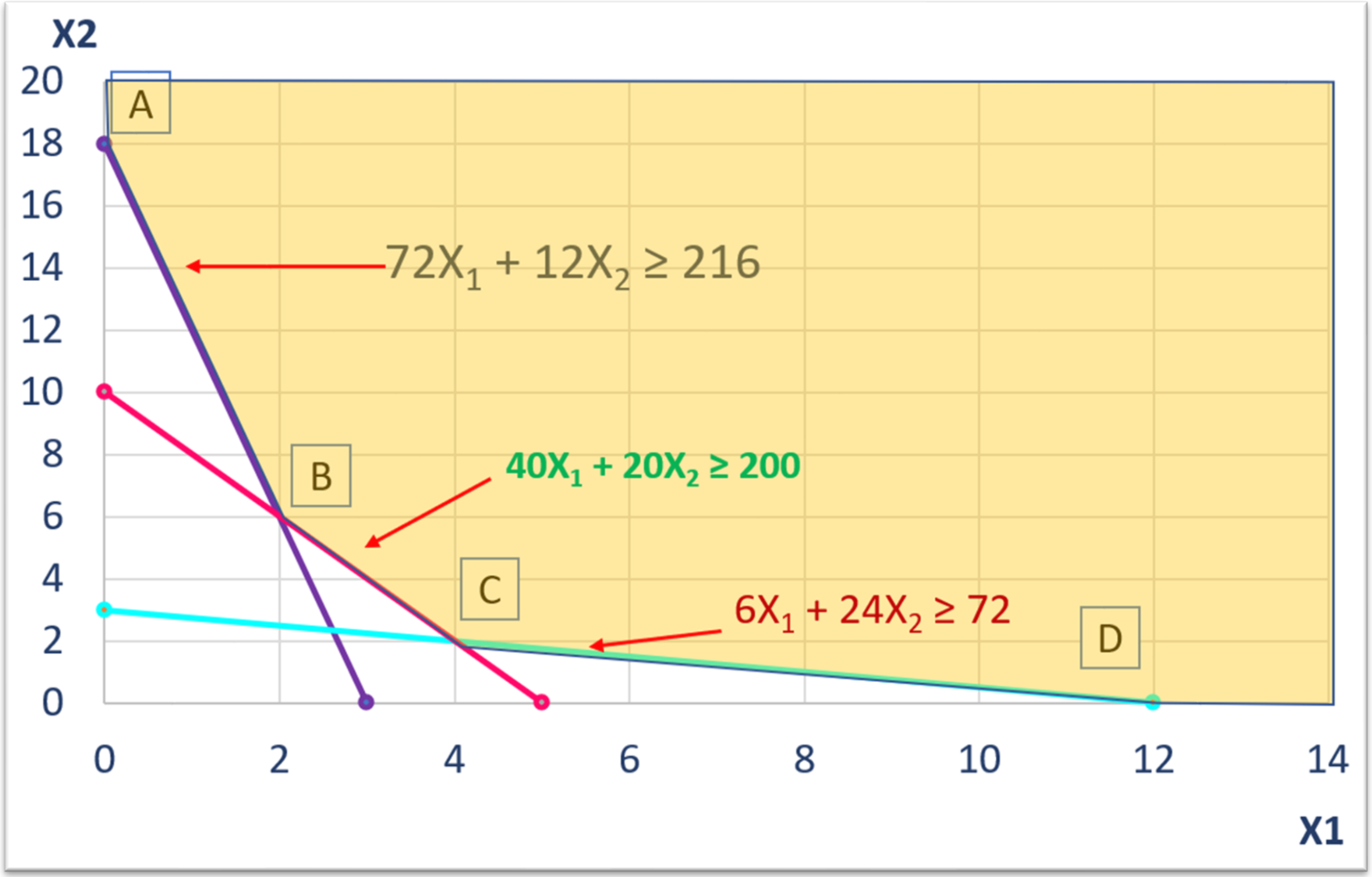
**Constraint No.3:**



# Methodology of Graphical Method

## Step 5: Identification of Feasible Region

- The final graph will look like:




All constraints are greater than or equal to (≥) type. Hence, feasible region should be above (to the right of) all constraints. The vertices of the feasible region are A,B,C & D.

# Methodology of Graphical Method

## Step 6: Finding the optimal solution

- The optimal 'solution always lies at one of the vertices or corners of the feasible region.
- To find optimal solution:
- We use corner point method. We find co-ordinates (X1 and X2 values) for each vertex or corner point. From this we find 'Z' value for each corner point.

| Corner point method: | | |
|---|---|---|
| **Vertex** | **Co-ordinates** | **$Z = 40X_1 + 80X_2$** |
| **A** | $X_1 = 0; X_2 = 18$ | Z =1440 |
| | From Graph | |
| **B** | $X_1 = 2, X_2 = 6$ | Z = Rs. 560 |
| | From simultaneous equations | |
| **C** | $X_1 = 4, X_2 = 2$ | Z = Rs.320 |
| | From simultaneous equations | |
| **D** | $X_1 = 12, X_2 = 0$ | Z = Rs. 480 |
| | From Graph | |

**Therefore, Min Z = Rs.320 (At point 'C')**

**Solution:**

Optimal cost = Min Z = Rs. 320

**Optimal Product Mix:**

$X_1$ = No. of units of product A= 4

$X_2$ = No. of units of product B = 2

# Methodology of Graphical Method

## Step 1: Formulation of LPP (Linear Programming Problem)

- Use the given data to formulate the LPP.

**Maximization-Mixed Constraints:**

**Example 3:** A firm makes two products $P_1$ & $P_2$ and has a production capacity of 18 tonnes per day. $P_1$ & $P_2$ require the same production capacity. The firm must supply at least 4 t of $P_1$ & 6 t of $P_2$ per day. Each tonne of $P_1$ & $P_2$ requires 60 hrs of machine work each. The maximum number of machine-hours available is 720. Profit per tonne for $P_1$ is Rs.160 and for $P_2$ is Rs. 240. Find the optimal solution by the graphical method.

# Methodology of Graphical Method

## Maximization-Mixed Constraints:

**Example 3:** A firm makes two products $P_1$ & $P_2$ and has a production capacity of 18 tonnes per day. $P_1$ & $P_2$ require the same production capacity. The firm must supply at least 4 t of $P_1$ & 6 t of $P_2$ per day. Each tonne of $P_1$ & $P_2$ requires 60 hrs of machine work each. The maximum number of machine-hours available is 720. Profit per tonne for $P_1$ is Rs.160 and for $P_2$ is Rs. 240. Find the optimal solution by the graphical method.

### Step 1: Formulation as LPP

$X_1$ = Tonnes of $P_1$/Day

$X_2$ = Tonnes of $P_2$/Day

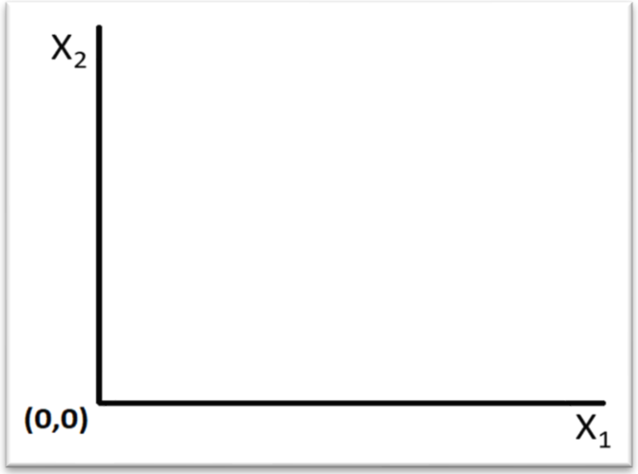
**Max. $Z = 160X_1 + 240X_2$**

**Subject to constraints:**

$$X_1 \geq 4$$
$$X_2 \geq 6$$
$$X_1 + X_2 \leq 18$$
$$60X_1 + 60X_2 \leq 720$$
$$X_1, X_2 \geq 0$$

### Step 2: Determination of each axis.

- Horizontal (X) axis: Product $P_1$ ($X_1$)
- Vertical (Y) axis: Product $P_2$ ($X_2$)



# Methodology of Graphical Method

Step 3: Finding coordinates of constraint lines to represent constraint lines on the graph.

- The constraints are present in the form of inequality (≤ or ≥). We should convert them into equality to obtain coordinates.

**Coordinates for constraint lines :**

(1) $X_1 \geq 4$ _ _ _ _ _ (4,0) _ _ _ _ _ No, value for $X_2$, Therefore, $X_2 = 0$

**(2)** $X_2 \geq 6$ _ _ _ _ _ (0,6) _ _ _ _ _ No, value for $X_1$, Therefore, $X_1 = 0$

**(3)** $X_1 + X_2 \leq 18$ _ _ _ _ _ (18,0), (0,18)

**(4)** $60X_1 + 60X_2 \leq 720$ _ _ _ _ _ (12,0), (0,12)

If, $X_1 = 0$, $60X_2 = 720$ _ _ _ _ Therefore, $X_2 = 12$ (0,12)

If, $X_2 = 0$, $60X_1 = 720$ _ _ _ _ Therefore, $X_1 = 12$ (12,0)

# Methodology of Graphical Method

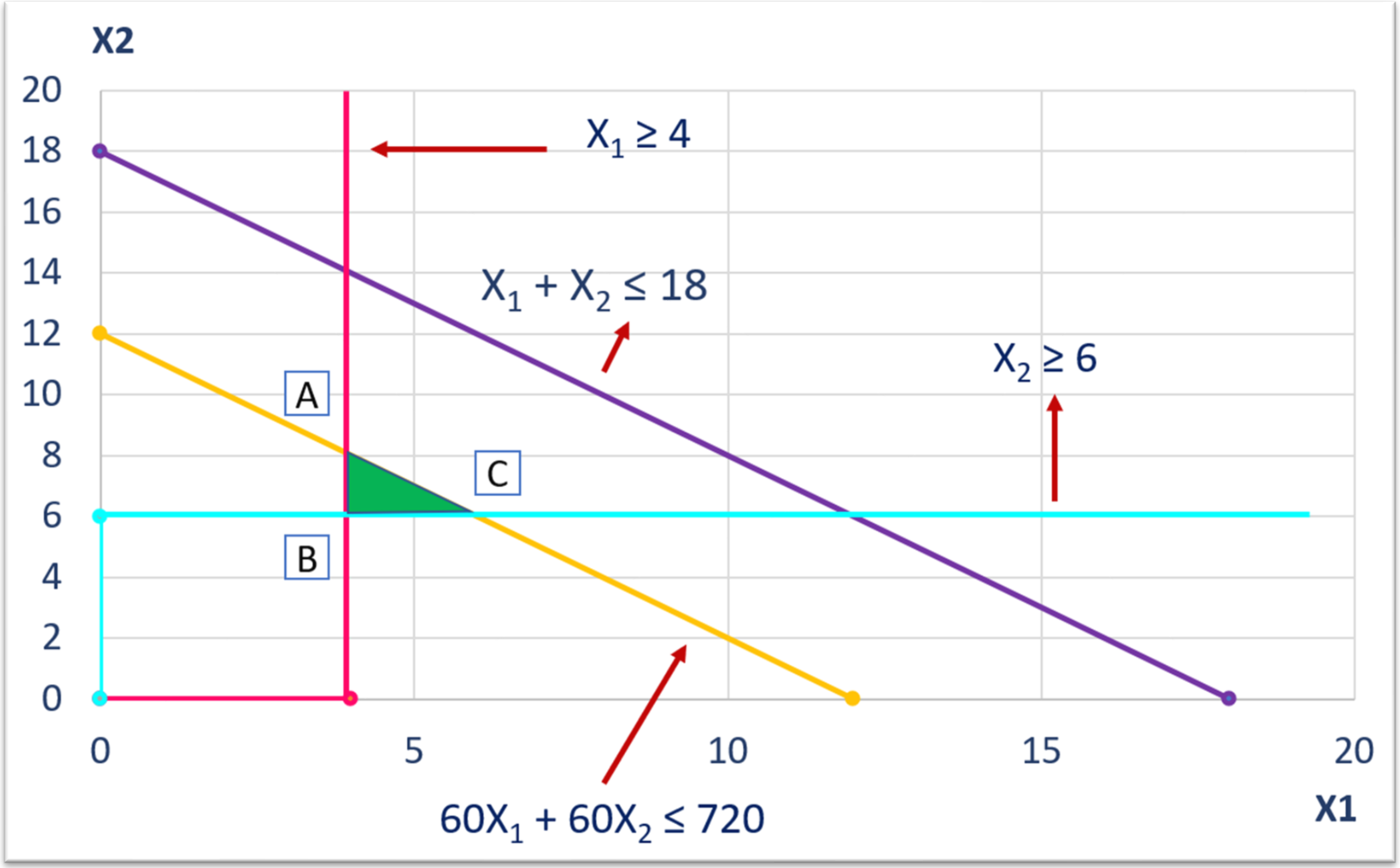
## Step 4: Representing constraint lines on the graph

- To mark the points on the graph, we need to select the appropriate scale.

To select scale:
Maximum value for $X_1$ = 18
Maximum value for $X_2$ = 18
Scale: 1 cm = 2 Tonnes

**Feasible Region: ABC**

Two constraints are 'greater than or equal to' type. Hence, the feasible region will be above or to the right of these constraint lines. Two constraints are 'less than or equal to' type. Hence, the feasible region will be below or to the left of these constraint lines. Hence, the feasible region is ABC.



# Methodology of Graphical Method

## Step 6: Finding the optimal solution

- The optimal 'solution always lies at one of the vertices or corners of the feasible region.

| Corner point method: | | |
|---|---|---|
| **Vertex** | **Co-ordinates** | **$Z = 160X_1 + 240X_2$** |
| **A** | $X_1 = 4; X_2 = 8$ | Z =Rs.2560 |
| | From simultaneous equations | |
| **B** | $X_1 = 4, X_2 = 6$ | Z = Rs. 2080 |
| | From Graph | |
| **C** | $X_1 = 6, X_2 = 6$ | Z = Rs.2400 |
| | From simultaneous equations | |

**For A→$X_1$ = 4 from graph**
A is on the line $60X_1 + 60X_2 = 720$
$60X_2 = 720 - 60\ (4) = 480$ _ _ _ _ Therefore, $X_2 = 8$
**For C → $X_2$ = 6 from graph**
C is on the line $60X_1 + 60X_2 = 720$
$60X_1 = 720 - 60\ (6) = 360$ _ _ _ _ Therefore, $X_1 = 6$
**Z max = Rs. 2,560 [At point 'A']**

**Max. Z = Rs. 2560 (At point A)**

**Solution:**
Optimal profit = Max Z = Rs. 2560
**Optimal Product Mix:**
$X_1$ = Production of $P_1$ = 4 Tonnes
$X_2$ = Tonnes of $P_2$ = 8 Tonnes

# Methodology of Graphical Method

## Step 1: Formulation of LPP (Linear Programming Problem)

- Use the given data to formulate the LPP.

**Minimization-Mixed Constraints:**

**Example 4:** A firm produces two products P and Q. Daily production upper limit is 600 units for total production. But at least 300 total units must be produced every day. Machine hours consumption per unit is 6 for P and 2 for Q. At least 1200 machine hours must be used daily. Manufacturing costs per unit are Rs. 50 for P and Rs. 20 for Q. Find the optimal solution for the LPP graphically.

# Methodology of Graphical Method

## Minimization-Mixed Constraints:

**Example 4:** A firm produces two products P and Q. Daily production upper limit is 600 units for total production. But at least 300 total units must be produced every day. Machine hours consumption per unit is 6 for P and 2 for Q. At least 1200 machine hours must be used daily. Manufacturing costs per unit are Rs. 50 for P and Rs. 20 for Q. Find the optimal solution for the LPP graphically.

**Step 1: Formulation as LPP**

$X_1$ = No. of Units of P per day

$X_2$ = No. of Units of Q per day

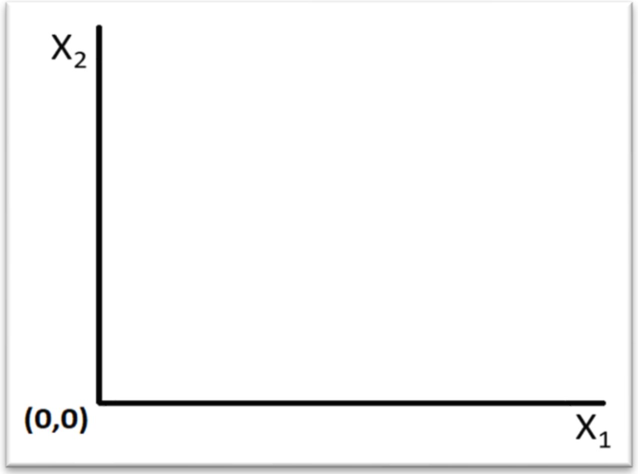
**Min. Z = 50$X_1$ + 20$X_2$**

**Subject to constraints:**

$$X_1 + X_2 \leq 600$$
$$X_1 + X_2 \geq 300$$
$$6X_1 + 2X_2 \geq 1200$$
$$X_1, X_2 \geq 0$$

**Step 2: Determination of each axis.**

- Horizontal (X) axis: Product P ($X_1$)
- Vertical (Y) axis: Product Q ($X_2$)



# Methodology of Graphical Method

Step 3: Finding coordinates of constraint lines to represent constraint lines on the graph.

- The constraints are present in the form of inequality (≤ or ≥). We should convert them into equality to obtain coordinates.

**Coordinates for constraint lines :**

**(1) $X_1 + X_2 = 600$ ……. Converting in equality**

If $X_1 = 0$, $X_2 = 600$, _ _ _ _ _ (0,600)

If $X_2 = 0$, $X_1 = 600$, _ _ _ _ _ (600,0)

**(2) $X_1 + X_2 = 300$ ……. Converting in equality**

If $X_1 = 0$, $X_2 = 300$, _ _ _ _ _ (0,300)

If $X_2 = 0$, $X_1 = 300$, _ _ _ _ _ (300,0)

**(3) $6X_1 + 2X_2 = 1200$……. Converting in equality**

If $X_1 = 0$, $2X_2 = 1200$, _ _ _ _ _ $X_2 = 600$…….. (0,600)

If $X_2 = 0$, $6X_1 = 1200$, _ _ _ _ _ $X_1 = 200$……..(200,0)

# Methodology of Graphical Method

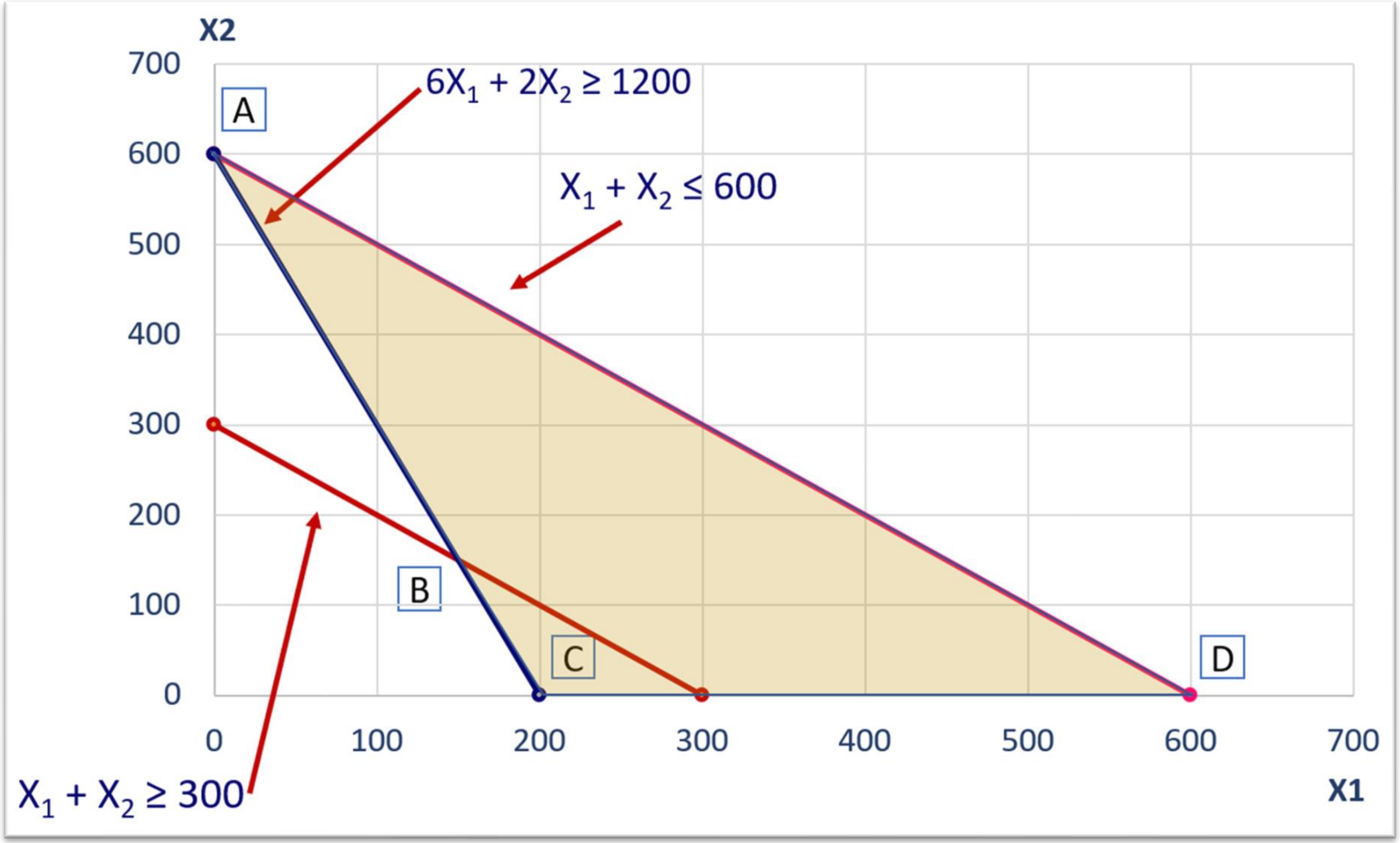
## Step 4: Representing constraint lines on the graph

- To mark the points on the graph, we need to select the appropriate scale.

To select scale:
Maximum value for $X_1$ = 600
Maximum value for $X_2$ = 600
Scale: 1 cm = 50 units

**Feasible Region: ABCD**

Two constraints are of 'greater than or equal to' type. Hence, the feasible region is to the right of or above these constraint lines. One constraint is of 'less than or equal to' type. Hence, the feasible region is to the left of or below that constraint line.



# Methodology of Graphical Method

## Step 6: Finding the optimal solution

- The optimal 'solution always lies at one of the vertices or corners of the feasible region.

| Corner point method: | | |
|---|---|---|
| **Vertex** | **Co-ordinates** | **$Z = 50X_1 + 20X_2$** |
| **A** | $X_1 = 0$; $X_2 = 600$ | Z =Rs.12,000 |
| | From Graph | |
| **B** | $X_1 = 150$, $X_2 = 150$ | Z = Rs. 10,500 |
| | From simultaneous equations | |
| **C** | $X_1 = 300$, $X_2 = 0$ | Z = Rs.15,000 |
| | From simultaneous equations | |
| **D** | $X_1 = 600$; $X_2 = 0$ | Z =Rs.30,000 |
| | From Graph | |

**For B→B is at intersection of two constraint lines**

**$6X_1 + 2X_2 = 1200$ and $X_1 + X_2 = 300$**

**$6X_1 + 2X_2 = 1200$ ......(1)**

**$X_1 + X_2 = 300$ .....(2)**

**$2X_1 + 2X_2 = 600$ .....(2) x 2**

**Therefore, $4X_1 = 600$**

**Therefore, $X_1 = 150$**

**Substituting value in equation (2), $X_2 = 150$**

**Min. Z = Rs. 10,500 (At point B)**

**Solution:**

Optimal cost = Min Z = Rs. 10,500

$X_1$ = No. of Units of P per day = 150

$X_2$ = No. of Units of Q per day = 150

# Methodology of Graphical Method

*LINEAR PROGRAMMING FOR TRAFFIC CONTROL AT HANSHIN EXPRESSWAY**

The Hanshin Expressway was the first urban toll expressway in Osaka, Japan. Although in 1964 its length was only 2.3 kilometers, today it is a large-scale urban expressway network of 200 kilometers. The Hanshin Expressway provides service for the Hanshin (Osaka-Kobe) area, the second-most populated area in Japan. An average of 828,000 vehicles use the expressway each day, with daily traffic sometimes exceeding 1 million vehicles. In 1990 the Hanshin Expressway Public Corporation started using an automated traffic control system in order to maximize the number of vehicles flowing into the expressway network.

The automated traffic control system relies on two control methods: (1) limiting the number of cars that enter the expressway at each entrance ramp; and (2) providing drivers with up-to-date and accurate traffic information, including expected travel times and information about accidents. The approach used to limit the number of vehicles depends upon whether the expressway is in a normal or steady state of operation, or whether some type of unusual event, such as an accident or a breakdown, has occurred.

In the first phase of the steady-state case, the Hanshin system uses a linear programming model to maximize the total number of vehicles entering the system, while preventing traffic congestion and adverse effects on surrounding road networks. The data that drive the linear programming model are collected from detectors installed every 500 meters along the expressway and at all entrance and exit ramps. Every five minutes the real-time data collected from the detectors are used to update the model coefficients, and a new linear program computes the maximum number of vehicles the expressway can accommodate.

The automated traffic control system proved successful. According to surveys, traffic control decreased the length of congested portions of the expressway by 30% and the duration by 20%. In addition to its extreme cost-effectiveness, drivers consider it an indispensable service.

*Based on T. Yoshino, T. Sasaki, and T. Hasegawa, "The Traffic-Control System on the Hanshin Expressway," *Interfaces* (January/February 1995): 94–108.

# Methodology of Graphical Method

**Example 5:** Print Well Pvt. Ltd is facing a tight financial squeeze and hence are attempting cost-saving wherever possible. The current contract is to print a book in hardcover and paperback. The cost of a hardcover type is Rs. 600 per 100 copies and Rs. 500 per 100 copies of paperback type. The company decides to run their two printing presses PI and PII for at least 80 hours and 60 hours respectively every week. PI can produce 100 hardcover books in 2 hours and 100 paperbacks in 1 hour. PII can produce 100 hardcover books in 1 hour and 100 paperbacks in 2 hours. Determine how many books of each type should be produced to minimize costs. Use the graphical method of linear programming.

# Methodology of Graphical Method

**Example 5:** Print Well Pvt. Ltd is facing a tight financial squeeze and hence are attempting cost-saving wherever possible. The current contract is to print a book in hardcover and paperback. The cost of a hardcover type is Rs. 600 per 100 copies and Rs. 500 per 100 copies of paperback type. The company decides to run their two printing presses PI and PII for at least 80 hours and 60 hours respectively every week. PI can produce 100 hardcover books in 2 hours and 100 paperbacks in 1 hour. PII can produce 100 hardcover books in 1 hour and 100 paperbacks in 2 hours. Determine how many books of each type should be produced to minimize costs. Use the graphical method of linear programming.

**Step 1: Formulation as LPP**

**Decision Variables:**

$X_1$ = No. of hardcover copies (in a lot of 100)

$X_2$ = No. of paperback copies (in a lot of 100)

**Min. $Z = 600X_1 + 500X_2$**

**Subject to constraints:?**

# Methodology of Graphical Method

**Example 5:** Print Well Pvt. Ltd is facing a tight financial squeeze and hence are attempting cost-saving wherever possible. The current contract is to print a book in hardcover and paperback. The cost of a hardcover type is Rs. 600 per 100 copies and Rs. 500 per 100 copies of paperback type. The company decides to run their two printing presses PI and PII for at least 80 hours and 60 hours respectively every week. PI can produce 100 hardcover books in 2 hours and 100 paperbacks in 1 hour. PII can produce 100 hardcover books in 1 hour and 100 paperbacks in 2 hours. Determine how many books of each type should be produced to minimize costs. Use the graphical method of linear programming.

**Constraints:**

| | Hardcover (in 100) | Paperback (in 100) | |
|---|---|---|---|
| PI (per 100) | 2 hrs | 1 hrs | At least 80 hrs. |
| PII (per 100) | 1 hrs | 2 hrs | At least 60 hrs. |

**Subject to constraints:**

$$2X_1 + 1X_2 \geq 80 \text{ .......PI}$$

$$1X_1 + 2X_2 \geq 60 \text{ .......PII}$$

$$X_1, X_2 \geq 0$$

# Methodology of Graphical Method

**Min. Z = $600X_1 + 500X_2$**
**Subject to constraints:**
$2X_1 + 1X_2 \geq 80$ …….PI
$1X_1 + 1X_2 \geq 60$ …….PII
$X_1, X_2 \geq 0$

**Coordinates for constraint lines :**

**(1) $2X_1 + 1X_2 = 80$ …….PI……. Converting in equality**

If $X_2 = 0, X_1 = 40$, _ _ _ _ _ (40,0)

If $X_1 = 0, X_2 = 80$, _ _ _ _ _ (0,80)

**(2) $1X_1 + 2X_2 = 60$ …….PII……. Converting in equality**

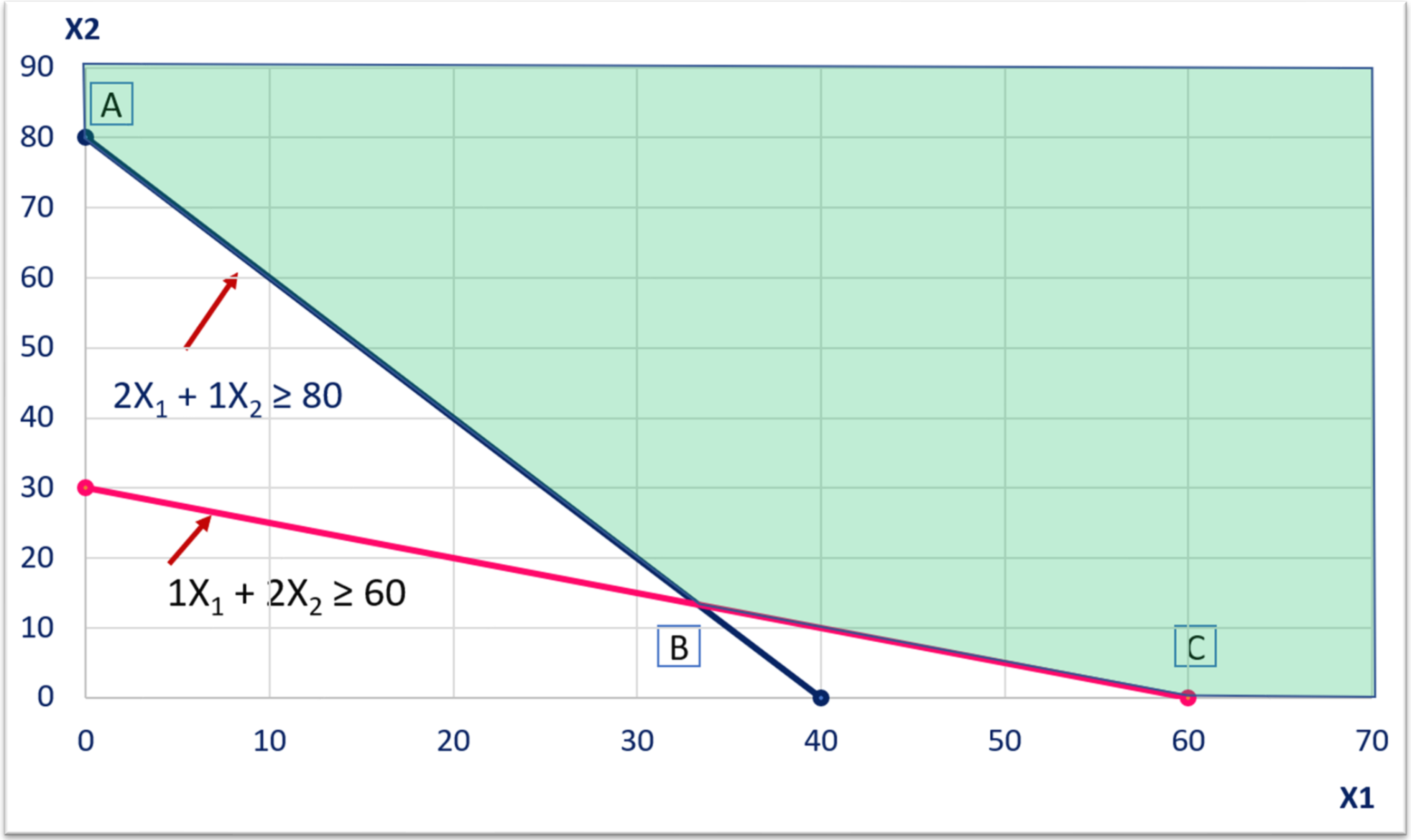
If $X_2 = 0, X_1 = 60$, _ _ _ _ _ (60,0)

If $X_1 = 0, X_2 = 30$, _ _ _ _ _ (0,30)

# Methodology of Graphical Method

To select scale:
Maximum value for $X_1$ = 60
Maximum value for $X_2$ = 80
Scale: 1 cm = 10 units

**Feasible Region: ABC**



# Methodology of Graphical Method

| Corner point method: | | |
|---|---|---|
| Vertex | Co-ordinates | $Z = 600X_1 + 500X_2$ |
| A | $X_1 = 0; X_2 = 80$ | Z =Rs.40,000 |
| | From Graph | |
| B | $X_1 = 100/3, X_2 = 40/3$ | Z = Rs. 80,000/3 |
| | From simultaneous equations | |
| C | $X_1 = 60, X_2 = 0$ | Z = Rs.36,000 |
| | From Graph | |

**Min. Z = Rs. 80,000/3 (At point B)**

**Solution:**
Optimal cost = Min Z = Rs. 80,000/3
$X_1$ = No. of hardcover copies (in a lot of 100) = 100/3
$X_2$ = No. of paperback copies (in a lot of 100) = 40/3

**For B→B is at intersection of two constraint lines**
**$2X_1 + 1X_2 = 80$ and $1X_1 + 2X_2 = 60$**
**$2X_1 + 1X_2 = 80$ ……(1)**
**$1X_1 + 2X_2 = 60$ …..(2)**

**$2X_1 + 4X_2 = 120$ …..(2) x 2…(3)**

**Therefore, $3X_2 = 40$ …..(3) - 2**

**Therefore, $X_2 = 40/3$**
**Substituting value in equation (1),**
**$2X_1 + (1*40/3) = 80$**
**$2X_1 = 80 - 40/3$**
**$2X_1 = 200/3$**
**$X_1 = 200/6$**
**$X_1 = 100/3$**

# Application of OT/OR: 3.0

## An Application Vignette

**GE Plastics Optimizes the Two-Echelon Global Fulfilment Network at Its High-Performance Polymers (HPP) Division**

"GE Plastics(GEP) is a $5 billion global materials business that supplies plastics and raw materials to various industries such as automotive, appliance, computer and medical equipment. GEP has set up manufacturing plants all over the globe. To remain competitive, GEP must deliver customized products to its customers anywhere in the world in the right quantity, at the right time, and at reasonable cost. With increasing supply chain complexity, the burden on GE decision systems that allocate global capacities is immense. In the past, GEP practiced a pole-centric manufacturing philosophy, making each product in the geographic area (Americas, Europe or Pacific) where it was to be delivered. Many of GEP's customers have since shifted their manufacturing operations to the Pacific. The result was a geographic imbalance between GEP's capacity and demand in the form of overcapacity in the Americas and under-capacity in the Pacific. Realizing that the pole-centric approach was no longer effective, in 2000 GEP adopted a global approach to its manufacturing operations. While GEP's primary motivation was to achieve a better balance between capacity and demand, it gained three additional opportunities to reduce operating costs: (1) to achieve economy of scale by centralizing production, (2) to reduce raw material costs by sourcing them globally and (3) to take advantage of tax breaks some countries offered it to set up and operate plants even if it had sufficient capacity in that region.

GEP developed a decision-support system (DSS) to optimize the two-echelon global manufacturing supply chain for its high-performance polymers (HPP) division. The DSS used a linear-programming model to maximize contribution margin while taking into consideration product demands and prices, plant capacities, production costs, distribution costs and raw material costs. The problem of optimizing the HPP supply chain can be stated as follows: given the market demand and price of HPP products, plant

# Application of OT/OR : 3.0

capacities, manufacturing costs, additive costs and distribution costs, determine the optimal production quantities at each plant line that will maximize the total contribution margin for the division. The results of the model are the optimal production quantities by plant and the total contribution margin. The DSS is implemented in Excel and uses LINGO to solve the optimization model.

The use of this linear programming-based DSS in GEP's high-performance polymers' (HPP), long-term planning was extremely well received and it was termed a best practice, and now other divisions with similar supply chains are adapting this tool for their own supply chain design and analysis. For HPP, this DSS is now a perpetual long-term design and short-term planning tool."

**Source:** Tyagi, Rajesh, Peter Kalish, Kunter Akbay, and Glenn Munshaw. 'GE Plastics optimizes the two-echelon global fulfillment network at its high performance polymers division'. Interfaces 34, no. 5 (2004): 359–366.

# Application of OT/OR : 3.1

## An Application Vignette

"**Swift & Company** is a diversified protein-producing business based in Greeley, Colorado. With annual sales of over $8 billion, beef and related products are by far the largest portion of the company's business.

To improve the company's sales and manufacturing performance, upper management concluded that it needed to achieve three major objectives. One was to enable the company's customer service representatives to talk to their more than 8,000 customers with accurate information about the availability of current and future inventory while considering requested delivery dates and maximum product age upon delivery. A second was to produce an efficient shift-level schedule for each plant over a 28-day horizon. A third was to accurately determine whether a plant can ship a requested order-line-item quantity on the requested date and time given the availability of cattle and constraints on the plant's capacity.

To meet these three challenges, an OR team developed an *integrated system of 45 linear programming models* based on three model formulations to dynamically schedule its beef-fabrication operations at five plants in real time as it receives orders. *The total audited benefits realized in the first year* of operation of this system *were* **$12.74 million,** including $12 million due to *optimizing the product mix*. Other benefits include a reduction in orders lost, a reduction in price discounting, and better on-time delivery."

**Source:** A. Bixby, B. Downs, and M. Self, "A Scheduling and Capable-to-Promise Application for Swift & Company," *Inter-faces*, **36**(1): 39–50, Jan.–Feb. 2006. (A link to this article is provided on our website, www.mhhe.com/hillier.)

# Application of OT/OR : 3.4

## An Application Vignette

"*Prostate cancer* is the most common form of cancer diagnosed in men. It is estimated that there were nearly 240,000 new cases and nearly 30,000 deaths in just the United States alone in 2013. Like many other forms of cancer, *radiation therapy* is a common method of treatment for prostate cancer, where the goal is to have a sufficiently high radiation dosage in the tumor region to kill the malignant cells while minimizing the radiation exposure to critical healthy structures near the tumor. This treatment can be applied through either *external beam* radiation therapy (as illustrated by the first example in this section) or *brachytherapy,* which involves placing approximately 100 radioactive "seeds" within the tumor region. The challenge is to determine the most effective three-dimensional geometric pattern for placing these seeds.

**Memorial Sloan-Kettering Cancer Center** (**MSKCC**) in New York City is the world's oldest private cancer center. An OR team from the *Center for Operations Research in Medicine and HealthCare* at Georgia Institute of Technology worked with physicians at MSKCC to develop a highly sophisticated *next-generation method* of optimizing the application of brachytherapy to prostrate cancer. The underlying model fits the structure for linear programming with one exception. In addition to having the usual continuous variables that fit linear programming, the model also has some *binary variables* (variables whose only possible values are 0 and 1). (This kind of extension of linear programming to what is called *mixed-integer programming* will be discussed in Chap. 12.) The optimization is done in a matter of minutes by an automated computerized planning system that can be operated readily by medical personnel when beginning the procedure of inserting the seeds into the patient's prostrate.

This breakthrough in optimizing the application of brachytherapy to prostrate cancer is having a profound impact on both health care costs and quality of life for treated patients because of its much greater effectiveness and the substantial reduction in side effects. When all U.S. clinics adopt this procedure, it is estimated that the annual cost savings will approximate **$500 million** due to eliminating the need for a pretreatment planning meeting and a postoperation CT scan, as well as providing a more efficient surgical procedure and reducing the need to treat subsequent side effects. It also is anticipated that this approach can be extended to other forms of brachytherapy, such as treatment of breast, cervix, esophagus, biliary tract, pancreas, head and neck, and eye.

This application of linear programming and its extensions led to the OR team winning the prestigious First Prize in the 2007 international competition for the Franz Edelman Award for Achievement in Operations Research and the Management Sciences."

**Source:** E. K. Lee and M. Zaider, "Operations Research Advances Cancer Therapeutics," *Interfaces,* **38**(1): 5–25, Jan.–Feb. 2008. (A link to this article is provided on our website, www.mhhe.com/hillier.)

# Application of OT/OR : 3.5

## An Application Vignette

"**Welch's, Inc.,** is the world's largest processor of Concord and Niagara grapes, with net sales of $650 million in 2012. Such products as Welch's grape jelly and Welch's grape juice have been enjoyed by generations of American consumers.

Every September, growers begin delivering grapes to processing plants that then press the raw grapes into juice. Time must pass before the grape juice is ready for conversion into finished jams, jellies, juices, and concentrates.

Deciding how to use the grape crop is a complex task given changing demand and uncertain crop quality and quantity. Typical decisions include what recipes to use for major product groups, the transfer of grape juice between plants, and the mode of transportation for these transfers.

Because Welch's lacked a formal system for optimizing raw material movement and the recipes used for production, an OR team developed a preliminary linear programming model. This was a large model with 8,000 decision variables that focused on the component level of detail. Small-scale testing proved that the model worked.

To make the model more useful, the team then revised it by aggregating demand by product group rather than by component. This reduced its size to 324 decision variables and 361 functional constraints. *The model then was incorporated into a spreadsheet.*

The company has run the continually updated version of this *spreadsheet model* each month since 1994 to provide senior management with information on the optimal logistics plan generated by the Solver. *The savings* from using and optimizing this model *were approximately* **$150,000** *in the first year alone.* A major advantage of incorporating the linear programming model into a spreadsheet has been the ease of explaining the model to managers with differing levels of mathematical understanding. This has led to a widespread appreciation of the operations research approach for both this application and others."

**Source:** E. W. Schuster and S. J. Allen, "Raw Material Management at Welch's, Inc.," *Interfaces,* **28**(5): 13–24, Sept.–Oct. 1998. (A link to this article is provided on our website, www.mhhe.com/hillier.)

# Application of OT/OR : 3.6

## An Application Vignette

"A key part of a country's financial infrastructure is its securities markets. By allowing a variety of financial institutions and their clients to trade stocks, bonds, and other financial securities, they securities markets help fund both public and private initiatives. Therefore, the efficient operation of its securities markets plays a crucial role in providing a platform for the economic growth of the country.

Each central securities depository and its system for quickly settling security transactions are part of the operational backbone of securities markets and a key component of financial system stability. In Mexico, an institution called **INDEVAL** provides both the central securities depository and its security settlement system for the entire country. This security settlement system uses electronic book entries, modifying cash and securities balances, for the various parties in the transactions.

The total value of the securities transactions the INDEVAL settles averages over **$250 billion** daily. This makes INDEVAL the main liquidity conduit for Mexico's entire financial sector. Therefore, it is extremely important that INDEVAL's system for clearing securities transactions be an exceptionally efficient one that maximizes the amount of cash that can be delivered almost instantaneously after the transactions. Because of past dissatisfaction with this system, INDEVAL's Board of Directors ordered a major study in 2005 to completely redesign the system.

Following more than 12,000 man-hours devoted to this redesign, the new system was successfully launched in November 2008. The core of the new system is a huge linear programming model that is applied many times daily to choose which of thousands of pending transactions should be settled immediately with the depositor's available balances. Linear programming is ideally suited for this application because huge models can be solved quickly to maximize the value of the transactions settled while taking into account the various relevant constraints.

This application of linear programming has substantially enhanced and strengthened the Mexican financial infrastructure by reducing its daily liquidity requirements by **$130 billion**. It also reduces the intraday financing costs for market participants by more than **$150 million** annually. This application led to INDEVAL winning the prestigious First Prize in the 2010 international competition for the Franz Edelman Award for Achievement in Operations Research and the Management Sciences."

**Source**: D. Muñoz, M. de Lascurain, O. Romeo-Hernandez, F. Solis, L. de los Santoz, A. Palacios-Brun, F. Herrería, and J. Villaseñor, "INDEVAL Develops a New Operating and Settlement System Using Operations Research," *Interfaces* 41, no. 1 (January-February 2011), pp. 8–17. (A link to this article is provided on our Web site, www.mhhe.com/hillier.)