OPERATIONS RESEARCH

Linear Programming – III
Simplex Method: Part I

Topic 3

ITM BUSINESS SCHOOL
OPERATIONS AND SUPPLY CHAIN MANAGEMENT
SEMESTER II
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Linear Programming - III

Simplex Method

Introduction

- In the graphical method, we first formulate the problem. and then we represent the constraint lines on a graph. From the constraint lines, we find the Feasible region of the solution.
- The optimal solution is at one of the corners of the Feasible region. The graphical method can be used only for a two-variable problem (X_1 & X_2).
- Another method. of solving an LPP is the simplex method. The simplex method is iterative. It involves several iterations of the simplex algorithm. It is a step-by-step progression from one feasible solution to another until we reach an optimal solution.
- It means in simplex method we keep on improving the solution in every step and finally reach optimal solution.

Simplex Method

Maximization:

Example 1:

Max. $Z = 100 X_1 + 80X_2$ ……….. (Product A & B)

Subject to constraints:

$6X_1 + 4X_2 \leq 7200$ ………… (Resource I)

$2X_1 + 4X_2 \leq 4000$ ………… (Resource II)

$X_1, X_2 \geq 0$

Find optimal solution by simplex method.

Formulation as LPP

Example 1:

Max. $Z = 100 X_1 + 80X_2$ ……… (Product A & B)

Subject to constraints:

$6X_1 + 4X_2 \leq 7200$ ………… (Resource I)

$2X_1 + 4X_2 \leq 4000$ ………… (Resource II)

$X_1, X_2 \geq 0$

Find optimal solution by simplex method.

Solution:

Step 1: Formulation as LPP

- If the formulation is not given, we should convert the given data in the LPP formulation.

Example 1:

Max. $Z = 100 X_1 + 80X_2$ ……… (Product A & B)

Subject to constraints:

$6X_1 + 4X_2 \leq 7200$ ………… (Resource I)

$2X_1 + 4X_2 \leq 4000$ ………… (Resource II)

$X_1, X_2 \geq 0$

Find optimal solution by simplex method.

Standard Form

Step 2: Converting the LPP formulation in standard form.

- Standard form means introducing slack variables in the LPP formulation.

Slack Variable: A slack variable represents the unutilized Capacity of a resource. A slack variable is represented by 'S' (S_1, S_2, etc.)

1st Constraint is.

$6X_1 + 4X_2 \leq 7200$ ………… (Resource I)

It means resource I has capacity of 7200 units.

If after production, some capacity remains unutilized, it will be represented by slack variable S_1.

Hence, now we can write constraint 1 as:

$6X_1 + 4X_2 + S_1 = 7200$

This means out of 7200 units available of resource I, ($6X_1 + 4X_2$) will be used for production and S_1 is the unutilized capacity, if any.

Example 1:

Max. $Z = 100 X_1 + 80X_2$ ……… (Product A & B)

Subject to constraints:

$6X_1 + 4X_2 \leq 7200$ ……… (Resource I)

$2X_1 + 4X_2 \leq 4000$ ……… (Resource II)

$X_1, X_2 \geq 0$

Find optimal solution by simplex method.

Standard Form

Step 2: Converting the LPP formulation in standard form.

- Standard form means introducing slack variables in the LPP formulation.

2nd constraint

$2X_1 + 4X_2 + S_2 = 4000$

Where S_2 is the unutilized capacity of 2nd resource, if any.

Since S_1 and S_2 are slack variables (which represent the unutilized capacity of resources), their profit coefficient is zero.

Standard Form

Example 1:

Max. $Z = 100 X_1 + 80X_2$ ……….. (Product A & B)

Subject to constraints:

$6X_1 + 4X_2 \leq 7200$ ……….. (Resource I)

$2X_1 + 4X_2 \leq 4000$ ……….. (Resource II)

$X_1, X_2 \geq 0$

Find optimal solution by simplex method.

Step 2: Converting the LPP formulation in standard form.

- Standard form means introducing slack variables in the LPP formulation.

Standard form:

Objective function:

Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$

Subject to constraints:

$6X_1 + 4X_2 + S_1 = 7200$

$2X_1 + 4X_2 + S_2 = 4000$

$X_1, X_2, S_1, S_2 \geq 0$

Initial Basic Feasible Solution: Table 1

Standard form:

Objective function:

Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$

Subject to constraints:

$6X_1 + 4X_2 + S_1 = 7200$

$2X_1 + 4X_2 + S_2 = 4000$

$X_1, X_2, S_1, S_2 \geq 0$

Step 3: Writing 1st simplex table (Initial Basic Feasible Solution)

- The structure of the simplex table is as given below:

C_j			100	80	0	0	**Replacement Ratio**
Basis							
c	x	b	X_1	X_2	S_1	S_2	b/K.C
0	S_1	7200	6	4	1	0	7200/6 = 1200
0	S_2	4000	2	4	0	1	4000/2 = 2000
Z_j			0	0	0	0	
$\Delta = C_j - Z_j$			100	80	0	0	

Initial Basic Feasible Solution: Table 1

Standard form:
Objective function:
Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$
Subject to constraints:
$6X_1 + 4X_2 + S_1 = 7200$
$2X_1 + 4X_2 + S_2 = 4000$
$X_1, X_2, S_1, S_2 \geq 0$

Step 3: Writing 1st simplex table (Initial Basic Feasible Solution)

- The structure of the simplex table is as given below:

C_j							**Replacement Ratio**
Basis							
c	x	b	X_1	X_2	S_1	S_2	
Z_j							
$\Delta = C_j - Z_j$							

(1) In standard form, have 4 variables X_1, X_2, S_1, S_2. Hence, we have 4 columns for X_1, X_2, S_1, S_2.
Similarly, there are 2 constraints, hence, we have two rows for the constraints.

(2) 'Basis' contains 3 columns: c, x & b.
x = basis variables (variables present in the basis)
b = solution values or quantity of basis variables.
c = contribution or profit of basis variables.

Initial Basic Feasible Solution: Table 1

Standard form:
Objective function:
Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$
Subject to constraints:
$6X_1 + 4X_2 + S_1 = 7200$
$2X_1 + 4X_2 + S_2 = 4000$
$X_1, X_2, S_1, S_2 \geq 0$

Step 3: Writing 1st simplex table (Initial Basic Feasible Solution)

- The structure of the simplex table is as given below:

C_j							**Replacement Ratio**
Basis							
c	x	b	X_1	X_2	S_1	S_2	
Z_j							
$\Delta = C_j - Z_j$							

(3) **C_j Row:** The row in simplex table which represents profit or contribution of each variable in the objective function.

(4) **Z_j Row:** The row in simplex table which represents the decrease in the value of the objective function, if 1 unit of that variable is brought in the solution.

(5) **($\Delta = C_j - Z_j$) Row:** The row in simplex table which represents the net increase in the objective function, if 1 unit of that variable is brought in the solution.

Initial Basic Feasible Solution: Table 1

Standard form:
Objective function:
Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$
Subject to constraints:
$6X_1 + 4X_2 + S_1 = 7200$
$2X_1 + 4X_2 + S_2 = 4000$
$X_1, X_2, S_1, S_2 \geq 0$

Step 3: Writing 1st simplex table (Initial Basic Feasible Solution)

- The structure of the simplex table is as given below:

C_j							**Replacement Ratio**
Basis							
c	x	b	X_1	X_2	S_1	S_2	
Z_j							
$\Delta = C_j - Z_j$							

(5) **($\Delta = C_j - Z_j$) Row:** The row in simplex table which represents the net increase in the objective function, if 1 unit of that variable is brought in the solution.

Hence, positive value of Δ indicates gain or increase in profit and negative value of Δ indicates decrease in profit or loss.
'Δ' is pronounced as 'delta'.

Test of optimality in simplex:
A simplex solution is optimal when there is no positive Δ ($C_j - Z_j$) value in the solution. All Δ values are either negative or zero.

Initial Basic Feasible Solution: Table 1

Standard form:
Objective function:
Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$
Subject to constraints:
$6X_1 + 4X_2 + S_1 = 7200$
$2X_1 + 4X_2 + S_2 = 4000$
$X_1, X_2, S_1, S_2 \geq 0$

Step 3: Writing 1st simplex table (Initial Basic Feasible Solution)

- The structure of the simplex table is as given below:

C_j							Replacement Ratio
Basis							
c	x	b	X_1	X_2	S_1	S_2	
Z_j							
$\Delta = C_j - Z_j$							

(6) Key column (Incoming variable):
The variable which has maximum positive $(C_j - Z_j)$ value, is called incoming variable for next table.
In the next simplex table, this variable will enter the basis and it will replace one of the existing basis variables.
Key column → Max Δ

(7) Key Row (Outgoing variable):
The variable which goes out of the solution in the next table. It is replaced by incoming variable in the basis.
To find Key Row, we need to find replacement ratios for all basis variables.
The formula of replacement ratio is (b column) + (key column).
Ratio = b/K.C.
Key Row → Min. + Ratio

Initial Basic Feasible Solution: Table 1

Standard form:
Objective function:
Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$
Subject to constraints:
$6X_1 + 4X_2 + S_1 = 7200$
$2X_1 + 4X_2 + S_2 = 4000$
$X_1, X_2, S_1, S_2 \geq 0$

Step 4: Initial Basic Feasible Solution:

- Find the solution.

C_j			100	80	0	0	Replacement Ratio
Basis							
c	x	b	X_1	X_2	S_1	S_2	
0	S_1	7200	6	4	1	0	
0	S_2	4000	2	4	0	1	
Z_j			0	0	0	0	
$\Delta = C_j - Z_j$			100	80	0	0	

(a) In the 1st Simplex table (Initial Feasible Solution) the basis variables are always slack variables. It is assumed that there is no production activity (X_1, = 0, X_2 = 0). Hence, all capacity is unutilized.
Hence, 'x' column is S_1, S_2.

Table-1

C_j			100	80	0	0	Replacement Ratio
Basis			K.C				
c	x	b	X_1	X_2	S_1	S_2	
0	S_1	7200/6	6/6	4/6	1/6	0/6	
0	S_2	4000	2	4	0	1	
Z_j			0	0	0	0	
$\Delta = C_j - Z_j$			100	80	0	0	

Key value = 6

Z =

x1 = 100×1 + 0×0 = 100

x2 = 100×2/3 + 0×8/3 = 200/3

S1 = 100×1/6 + 0×-1/3 = 100/6 = 50/3

S2 = 100×0 + 0×1 = 0

2

C_j			100	80 (K.C.)	0	0	Replacement Ratio (min)
Basis							
c	x	b	X_1	X_2	S_1	S_2	
100	X1	1200	1	2/3	1/6	0	1200 × 2/3 = 800
0	S2	1600	0	8/3	-1/3	1	1600 × 8/3 = 600
Z_j			100	200/3	50/3	0	
$\Delta = C_j - Z_j$ = 0 or -ve			0	40/3	-50/3	0	

Table 2

C_j			100	80	0	0	Replacement Ratio
Basis							
c	x	b	X_1	X_2	S_1	S_2	
100	X_1	1200	1	2/3	1/6	0	1200*3/2 = 1800
0	S_2	1600	0	8/3	-1/3	1	1600*3/8 = 600
Z_j			100	200/3	50/3	0	
$\Delta = C_j - Z_j$			0	40/3	-50/3	0	

$1200 - \frac{2}{3} \times 600 = 800$

$1 - \frac{2}{3} \times 0 = 1$

$\frac{2}{3} - \frac{2}{3} \times 1 = 0$

$\frac{1}{6} - \frac{2}{3} \times -\frac{1}{8} = \frac{1}{4}$

$0 - \frac{2}{3} \times \frac{3}{8} = -\frac{1}{4}$

Table-3

Part 2

C_j			100	80	0	0	Replacement Ratio
Basis							
c	x	b	X_1	X_2	S_1	S_2	
100	x_1	800	1	0	1/4	-1/4	stop
80	x_2	600	0	1	-1/8	3/8	
Z_j			100	80	15	5	
$\Delta = C_j - Z_j$ → 0 or -ve			0	0	-15	-5	

Σ
rule: +ve

Initial Basic Feasible Solution: Table 1

Standard form:
Objective function:
Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$
Subject to constraints:
$6X_1 + 4X_2 + S_1 = 7200$
$2X_1 + 4X_2 + S_2 = 4000$
$X_1, X_2, S_1, S_2 \geq 0$

Step 4: Initial Basic Feasible Solution:

- Find the solution.

C_j			100	80	0	0	Replacement Ratio
Basis							
c	x	b	X_1	X_2	S_1	S_2	
0	S_1	7200	6	4	1	0	
0	S_2	4000	2	4	0	1	
Z_j							
$\Delta = C_j - Z_j$							

(b) Constraints:
$6X_1 + 4X_2 + \mathbf{S_1 = 7200}$
$2X_1 + 4X_2 + \mathbf{S_2 = 4000}$
Since all capacities are unutilized, hence $S_1 = 7200$ and $S_2 = 4000$
Hence, the 'b' column values are 7200 and 4000.

Standard form:
Objective function:
Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$
Subject to constraints:
$6X_1 + 4X_2 + S_1 = 7200$
$2X_1 + 4X_2 + S_2 = 4000$
$X_1, X_2, S_1, S_2 \geq 0$

Initial Basic Feasible Solution: Table 1

Step 4: Initial Basic Feasible Solution:

- Find the solution.

C_j			100	80	0	0	Replacement Ratio
Basis							
c	x	b	X_1	X_2	S_1	S_2	
0	S_1	7200	6	4	1	0	
0	S_2	4000	2	4	0	1	
Z_j							
$\Delta = C_j - Z_j$							

(c) Profit of slack variables is zero, hence 'C' column is 0,0.

Standard form:
Objective function:
Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$
Subject to constraints:
$6X_1 + 4X_2 + S_1 = 7200$
$2X_1 + 4X_2 + S_2 = 4000$
$X_1, X_2, S_1, S_2 \geq 0$

Initial Basic Feasible Solution: Table 1

Step 4: Initial Basic Feasible Solution:

- Find the solution.

C_j			100	80	0	0	Replacement Ratio
Basis							
c	x	b	X_1	X_2	S_1	S_2	
0	S_1	7200	6	4	1	0	
0	S_2	4000	2	4	0	1	
Z_j							
$\Delta = C_j - Z_j$							

(d) The C_j Row is written as per objective function.
Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$
Hence, C_j row values are 100,80,0, 0.

Standard form:
Objective function:
Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$
Subject to constraints:
$6X_1 + 4X_2 + S_1 = 7200$
$2X_1 + 4X_2 + S_2 = 4000$
$X_1, X_2, S_1, S_2 \geq 0$

Initial Basic Feasible Solution: Table 1

Step 4: Initial Basic Feasible Solution:

- Find the solution.

C_j			100	80	0	0	Replacement Ratio
Basis							
c	x	b	X_1	X_2	S_1	S_2	
0	S_1	7200	6	4	1	0	
0	S_2	4000	2	4	0	1	
Z_j							
$\Delta = C_j - Z_j$							

(e) 1st constraint is:

$6X_1 + 4X_2 + 1S_1 = 7200$

Hence, coefficients of X_1, X_2, S_1, S_2 are 6, 4, 1,0 (values in 1^{st} row)

Initial Basic Feasible Solution: Table 1

Standard form:
Objective function:
Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$
Subject to constraints:
$6X_1 + 4X_2 + S_1 = 7200$
$2X_1 + 4X_2 + S_2 = 4000$
$X_1, X_2, S_1, S_2 \geq 0$

Step 4: Initial Basic Feasible Solution:

- Find the solution.

C_j			100	80	0	0	Replacement Ratio
Basis							
c	x	b	X_1	X_2	S_1	S_2	
0	S_1	7200	6	4	1	0	
0	S_2	4000	2	4	0	1	
Z_j							
$\Delta = C_j - Z_j$							

(e) 2nd constraint is:
$2X_1 + 4X_2 + \mathbf{S_2 = 4000}$
Hence, coefficients of X_1, X_2, S_1, S_2 are 2, 4, 0, 1 (values in 2nd row).

Initial Basic Feasible Solution: Table 1

Standard form:
Objective function:
Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$
Subject to constraints:
$6X_1 + 4X_2 + S_1 = 7200$
$2X_1 + 4X_2 + S_2 = 4000$
$X_1, X_2, S_1, S_2 \geq 0$

Step 4: Initial Basic Feasible Solution:

- Find the solution.

C_j			100	80	0	0	Replacement Ratio
Basis							
c	x	b	X_1	X_2	S_1	S_2	
0	S_1	7200	6	4	1	0	
0	S_2	4000	2	4	0	1	
Z_j							
$\Delta = C_j - Z_j$							

(f) Calculation for Z_j Row:
Z_j = ['C' column] x [Each Variable column]

For X_1: $(0 \times 6) + (0 \times 2) = 0$
For X_2: $(0 \times 4) + (0 \times 4) = 0$
For S_1: $(0 \times 1) + (0 \times 0) = 0$
For S_2: $(0 \times 0) + (0 \times 1) = 0$

Initial Basic Feasible Solution: Table 1

Standard form:
Objective function:
Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$
Subject to constraints:
$6X_1 + 4X_2 + S_1 = 7200$
$2X_1 + 4X_2 + S_2 = 4000$
$X_1, X_2, S_1, S_2 \geq 0$

Step 4: Initial Basic Feasible Solution:

- Find the solution.

C_j			100	80	0	0	Replacement Ratio
Basis							
c	x	b	X_1	X_2	S_1	S_2	
0	S_1	7200	6	4	1	0	
0	S_2	4000	2	4	0	1	
Z_j							
$\Delta = C_j - Z_j$							

(g) Calculation for $\Delta = C_j - Z_j$ Row:
From the C_j value of each column, subtract Z_j value.]

Δ of $X_1 = 100 – 0 = 100$
Δ of $X_2 = 80 – 0 = 80$
Δ of $S_1 = 0 – 0 = 0$
Δ of $S_2 = 0 – 0 = 0$

Initial Basic Feasible Solution: Table 1

Standard form:
Objective function:
Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$
Subject to constraints:
$6X_1 + 4X_2 + S_1 = 7200$
$2X_1 + 4X_2 + S_2 = 4000$
$X_1, X_2, S_1, S_2 \geq 0$

C_j			100	80	0	0	Replacement Ratio
Basis							
c	x	b	X_1	X_2	S_1	S_2	
0	S_1	7200	6	4	1	0	7200/6 = 1200
0	S_2	4000	2	4	0	1	4000/2 = 2000
Z_j			0	0	0	0	
$\Delta = C_j - Z_j$			100	80	0	0	

After calculating all Δ values, we can see that maximum positive Δ value is for variable X_1.
Hence, Key Columns = X_1
It means in next simplex table; X_1 will enter the solution.

Initial Basic Feasible Solution: Table 1

Standard form:
Objective function:
Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$
Subject to constraints:
$6X_1 + 4X_2 + S_1 = 7200$
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$X_1, X_2, S_1, S_2 \geq 0$

C_j			100	80	0	0	Replacement Ratio
Basis							
c	x	b	X_1	X_2	S_1	S_2	
0	S_1	7200	6	4	1	0	7200/6 = 1200
0	S_2	4000	2	4	0	1	4000/2 = 2000
Z_j			0	0	0	0	
$\Delta = C_j - Z_j$			100	80	0	0	

(h) Calculation for Replacement Ratio:

Replacement ratio = 'b' column/key column

Hence, RR for S_1 = 7200/6 = 1200

RR for S_2 = 4000/2 = 2000

Minimum positive ratio is 1200 for S_1

Hence, **Key Row = S_1**

Initial Basic Feasible Solution: Table 1

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Subject to constraints:
$6X_1 + 4X_2 + S_1 = 7200$
$2X_1 + 4X_2 + S_2 = 4000$
$X_1, X_2, S_1, S_2 \geq 0$

C_j			100	80	0	0	Replacement Ratio
Basis							
c	x	b	X_1	X_2	S_1	S_2	
0	S_1	7200	6	4	1	0	7200/6 = 1200
0	S_2	4000	2	4	0	1	4000/2 = 2000
Z_j			0	0	0	0	
$\Delta = C_j - Z_j$			100	80	0	0	

It means in 2nd simplex table, in the Basis of the solution, S_1 will be replaced by X_1.

Key Element = 6, Intersection of X_1 and S_1

So, the 'X' column of 2nd Simplex table will be X_1 and S_2.

Now we can write 2nd Simplex table.

Standard form:
Objective function:
Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$
Subject to constraints:
$6X_1 + 4X_2 + S_1 = 7200$
$2X_1 + 4X_2 + S_2 = 4000$
$X_1, X_{2,} S_1, S_2 \geq 0$

Table 2

C_j			100	80	0	0	Replacement Ratio
Basis							
c	x	b	X_1	X_2	S_1	S_2	
100	X_1	1200	1	2/3	1/6	0	
0	S_2	1600	0	8/3	-1/3	1	
Z_j			100	200/3	50/3	0	
$\Delta = C_j - Z_j$			0	40/3	-50/3	0	

After calculating all Δ values, we can see that maximum positive Δ value is for variable X_1.
Hence, Key Columns = X_1
It means in next simplex table; X_1 will enter the solution.

Standard form:
Objective function:
Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$
Subject to constraints:
$6X_1 + 4X_2 + S_1 = 7200$
$2X_1 + 4X_2 + S_2 = 4000$
$X_1, X_2, S_1, S_2 \geq 0$

Table 2

C_j			100	80	0	0	Replacement Ratio
Basis							
c	x	b	X_1	X_2	S_1	S_2	
100	X_1	1200	1	2/3	1/6	0	
0	S_2	1600	0	8/3	-1/3	1	
Z_j			100	200/3	50/3	0	
$\Delta = C_j - Z_j$			0	40/3	-50/3	0	

Calculations of New values for 2nd table:
1st calculations: New values for key row:

New values = Old values/Key element

To find new values, we divide old values by key element.
Our key row for 1st table = S_1
Our key element for 1st = 6

Old values of S_1/key element. (7200/6 = 1200, 6/6=1, 4/6=2/3, 0/6=0)

These are the new values for X_1 row. Because X_1 has replaced S_1.

Standard form:
Objective function:
Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$
Subject to constraints:
$6X_1 + 4X_2 + S_1 = 7200$
$2X_1 + 4X_2 + S_2 = 4000$
$X_1, X_2, S_1, S_2 \geq 0$

Table 2

C_j			100	80	0	0	Replacement Ratio
Basis							
c	x	b	X_1	X_2	S_1	S_2	
100	X_1	1200	1	2/3	1/6	0	
0	S_2	1600	0	8/3	-1/3	1	
Z_j			100	200/3	50/3	0	
$\Delta = C_j - Z_j$			0	40/3	-50/3	0	

2nd calculation: new values for non-key row.

New values = Old values – corresponding key column value * new values of key row)

Non-key row S_2,
for S_2,
Corresponding key column value = 2…. in table 3
New values of S_2 = **Old values - corresponding key column value * new values of key row)**
(4000 – 2*1200 = 1600),
(2 – 2*1 = 0),
(4 – 2*2/3 = 8/3),
(0 – 2*1/6 = -1/3),
(1 – 2*0 = 1),

Standard form:

Objective function:

Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$

Subject to constraints:

$6X_1 + 4X_2 + S_1 = 7200$

$2X_1 + 4X_2 + S_2 = 4000$

$X_1, X_2, S_1, S_2 \geq 0$

Table 2

C_j			100	80	0	0	Replacement Ratio
Basis							
c	x	b	X_1	X_2	S_1	S_2	
100	X_1	1200	1	2/3	1/6	0	
0	S_2	1600	0	8/3	-1/3	1	
Z_j			100	200/3	50/3	0	
$\Delta = C_j - Z_j$			0	40/3	-50/3	0	

Calculation of Z_j:

for X_1: (100*1) + (0*0) = 100

for X_2: (100*2/3) + (0*8/3) = 200/3

for S_1: 100*1/6) + (0*-1/3) = 100/6=50/3

for S_2: (100*0) + (0*1) = 0

Table 2

Standard form:
Objective function:
Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$
Subject to constraints:
$6X_1 + 4X_2 + S_1 = 7200$
$2X_1 + 4X_2 + S_2 = 4000$
$X_1, X_2, S_1, S_2 \geq 0$

C_j			100	80	0	0	Replacement Ratio
Basis							
c	x	b	X_1	X_2	S_1	S_2	
100	X_1	1200	1	2/3	1/6	0	1200*3/2 = 1800
0	S_2	1600	0	8/3	-1/3	1	1600*3/8 = 600
Z_j			100	200/3	50/3	0	
$\Delta = C_j - Z_j$			0	40/3	-50/3	0	

Standard form:
Objective function:
Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$
Subject to constraints:
$6X_1 + 4X_2 + S_1 = 7200$
$2X_1 + 4X_2 + S_2 = 4000$
$X_1, X_2, S_1, S_2 \geq 0$

Table 2

C_j			100	80	0	0	Replacement Ratio
Basis							
c	x	b	X_1	X_2	S_1	S_2	
100	X_1	1200	1	2/3	1/6	0	1200*3/2 = 1800
0	S_2	1600	0	8/3	-1/3	1	1600*3/8 = 600
Z_j			100	200/3	50/3	0	
$\Delta = C_j - Z_j$			0	40/3	-50/3	0	

Key column: Max. + Δ = 40/3

Therefore, Key column = X_2

Replacement ratio = 'b' column/key column = b/ X_2
Ratio for X_1 = 1200/2/3 = 1200*3/2 = 1800

Ratio for S_2 = 1600/8/3 = 1600*8/3 = 600

Key row: Min. + Ratio = 600

Therefore, Key row = S_2
Key element = Intersection of X_2 & S_2 = 8/3

3^{rd} Simplex Table

Standard form:
Objective function:
Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$
Subject to constraints:
$6X_1 + 4X_2 + S_1 = 7200$
$2X_1 + 4X_2 + S_2 = 4000$
$X_1, X_2, S_1, S_2 \geq 0$

3^{rd} Simplex Table:

C_j			100	80	0	0	Replacement Ratio
Basis							
c	x	b	X_1	X_2	S_1	S_2	
100	X1	800	1	0	1/4	-1/4	
80	X2	600	0	1	-1/8	3/8	
Z_j			100	80	15	5	
$\Delta = C_j - Z_j$			0	0	-15	-5	

Now the basis variables in 3rd Simplex table will be X_1 and X_2. (X_2 replaces S_2)
The corresponding 'c' values are '100' for X_1 and '80' for X_2.

3^{rd} Simplex Table

Standard form:
Objective function:
Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$
Subject to constraints:
$6X_1 + 4X_2 + S_1 = 7200$
$2X_1 + 4X_2 + S_2 = 4000$
$X_1, X_2, S_1, S_2 \geq 0$

3^{rd} Simplex Table:

C_j			100	80	0	0
Basis						
c	x	b	X_1	X_2	S_1	S_2
100	X1	800	1	0	1/4	-1/4
80	X2	600	0	1	-1/8	3/8
Z_j			100	80	15	5
$\Delta = C_j - Z_j$			0	0	-15	-5

Calculations of new values for 3^{rd} table:

New values for key row:

For X_2, because X_2 has replaced S_2

New values = old values of S_2/key element

Therefore, new values = 1600

New values = 1600/8/3 = 600, 8/3/8/3=1, -1/3/8/3=-1/8, 1/8/3=3/8
These are new values of X_2 row.

3^{rd} Simplex Table

Standard form:
Objective function:
Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$
Subject to constraints:
$6X_1 + 4X_2 + S_1 = 7200$
$2X_1 + 4X_2 + S_2 = 4000$
$X_1, X_2, S_1, S_2 \geq 0$

3^{rd} Simplex Table:

C_j			100	80	0	0
Basis						
c	x	b	X_1	X_2	S_1	S_2
100	X1	800	1	0	1/4	-1/4
80	X2	600	0	1	-1/8	3/8
Z_j			100	80	15	5
$\Delta = C_j - Z_j$			0	0	-15	-5

New values for non-key row X_1:

Old values - corresponding key column value * new values of key row

1200 – (2/3*600) = 800

1 – (2/3*0) = 1

2/3 – (2/3*1) = 0

1/6 – (2/3*-1/8) = ¼

0 – (2/3*3/8) = - ¼

For X1 corresponding key column value = 2/3

New values of X1 are 800, 1, 0, ¼, -1/4

3rd Simplex Table

Standard form:
Objective function:
Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$
Subject to constraints:
$6X_1 + 4X_2 + S_1 = 7200$
$2X_1 + 4X_2 + S_2 = 4000$
$X_1, X_2, S_1, S_2 \geq 0$

3rd Simplex Table:

C_j			100	80	0	0
Basis						
c	x	b	X_1	X_2	S_1	S_2
100	X1	800	1	0	1/4	-1/4
80	X2	600	0	1	-1/8	3/8
Z_j			100	80	15	5
$\Delta = C_j - Z_j$			0	0	-15	-5

Calculation of Z_j:

for X_1: (100*1) + (80*0) = 100

for X_2: (100*0) + (80*1) = 80

for S_1: 100*1/4) + (80*-1/8) = 15

for S_2: (100*-1/4) + (80*3/8) = 5

3^{rd} Simplex Table

Standard form:
Objective function:
Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$
Subject to constraints:
$6X_1 + 4X_2 + S_1 = 7200$
$2X_1 + 4X_2 + S_2 = 4000$
$X_1, X_2, S_1, S_2 \geq 0$

3^{rd} Simplex Table:

C_j			100	80	0	0
Basis						
c	x	b	X_1	X_2	S_1	S_2
100	X1	800	1	0	1/4	-1/4
80	X2	600	0	1	-1/8	3/8
Z_j			100	80	15	5
$\Delta = C_j - Z_j$			0	0	-15	-5

Calculation of Δ:

For X_1: 100 - 100 = 0

For X_2: 80 - 80 = 0

For S_1 :0 -15 = -15

For S_2: 0 - 5 = -5

Optimal Solution

Standard form:

Objective function:

Max. $Z = 100X_1 + 80X_2 + 0S_1 + 0S_2$

Subject to constraints:

$6X_1 + 4X_2 + S_1 = 7200$

$2X_1 + 4X_2 + S_2 = 4000$

$X_1, X_2, S_1, S_2 \geq 0$

3rd Simplex Table:

C_j			100	80	0	0
Basis						
c	x	b	X_1	X_2	S_1	S_2
100	X1	800	1	0	1/4	-1/4
80	X2	600	0	1	-1/8	3/8
Z_j			100	80	15	5
$\Delta = C_j - Z_j$			0	0	-15	-5

All C_j - Z_j; values are either zero or negative. There is no positive **Δ**. Hence, solution is optimal.

riables					
B					
600					
			LHS		RHS
4	1	0	7200	≤	7200
4	0	1	4000	≤	4000
80			128000		
			Max. Z		

Optimal Solution:

Optimal product mix: ('b' values)

X_1 = No. of units of A = 800

X_2 = No. of units of B = 600

Optimal profit: 'c'*'b' values

Therefore, Max. Z = (100*800) * (80*600)

= Rs. 1,28,000

Optimal Solution

Note:

(1) 'Optimal Product Mix' means the solution values i.e., quantity values of variables.
For basis variables, quantity values are → X_1 = 800 and X_2 = 600.
For non-basis variables (variables which are not in the basis), quantity values are always zero.
S_1 and S_2 are not in the basis.
Hence, S_1 and S_2 are non-basis variables.
$S_1 = 0$ and $S_2 = 0$

(2) 'Optimal Profit' can be calculated by multiplying the. quantity values of basis variables ('b' column) by their respective profit coefficients ('C' column).
Therefore, Optimal Profit = 'C' column * 'b' column

Optimal Solution

(3) Concept of Shadow price of resources:

Shadow price of resource means value of one extra unit of resource. It is the maximum price the company should pay for procuring extra resources from market. It also. indicates profitability or profit contribution of each resource (per unit).

Shadow price = 'Z_j' value of slack variables

Z_j value of S_1= 15

Z_j value of S_2 = 5

S_1 is slack of Resource I and S_2 is slack of Resource II.

Shadow price of Resource I - Rs. 15/ unit

Shadow price of Resource II = Rs. 5/ unit

Optimal Solution

Availability of Resource I is 7200 units.
Hence, profit generated by Resource I = 7200 x 15 = Rs. 108,000

Availability of Resource II is 4000 units.
Hence, profit generated by Resource II = 4000 x 5 = Rs. 20,000.

Therefore, Total profit = 108,000 + 20,000 = Rs. 128,000

This corresponds with our Optimal profit calculated in Optimal solution.

Simplex Method

Minimization:

Example 2:

The Agricultural Research Institute suggested to a farmer to spread out at least 4,800 kg of a special phosphate fertiliser and not less than 7,200 kg of a special nitrogen fertiliser to raise productivity of crops in his fields. There are two sources for obtaining these—mixtures A and B. Both of these are available in bags weighing 100 kg each and they cost Rs. 40 and Rs. 24, respectively. Mixture A contains phosphate and nitrogen equivalent of 20 kg and 80 kg, respectively, while mixture B contains these ingredients equivalent of 50 kg each.

Write this as a linear programming problem to determine how many bags of each type the farmer should buy in order to obtain the required fertiliser at minimum cost.

LPP Formulation

Minimization:

The objective function: In this problem, such a combination of mixtures A and B is sought to be purchased as would minimize the total cost. If X_1 and X_2 are taken to represent the number of bags of mixtures A and B, respectively, the objective function can be expressed as follows:

Minimize $Z = 40X_1 + 24X_2$ ……………Cost

LPP Formulation

Minimization:

The constraints: Here, there are two constraints, namely, a minimum of 4,800 kg of phosphate and 7,200 kg of nitrogen ingredients are required. It is known that each bag of mixture A contains 20 kg and each bag of mixture B contains 50 kg of phosphate.

The phosphate requirement can be expressed as

$$\mathbf{20X_1 + 50X_2 \geq 4{,}800.}$$

Similarly, with the given information on the contents, the nitrogen requirement would be written as

$$\mathbf{80X_1 + 50\ X_2 \geq 7{,}200.}$$

LPP Formulation

Minimization:

The constraints: Here, there are two constraints, namely, a minimum of 4,800 kg of phosphate and 7,200 kg of nitrogen ingredients are required. It is known that each bag of mixture A contains 20 kg and each bag of mixture B contains 50 kg of phosphate.

The phosphate requirement can be expressed as $20X_1 + 50X_2 \geq 4{,}800$. Similarly, with the given information on the contents, the nitrogen requirement would be written as $80X_1 + 50\,X_2 \geq 7{,}200$.

Non-negativity condition: As before, it lays that the decision variables, representing the number of bags of mixtures A and B to use, would be non-negative. Thus, $X_1 \geq 0$ and $X_2 \geq 0$.

LPP Formulation

Minimization:

The linear programming problem can now be expressed as follows:

Minimize $Z = 40X_1 + 24X_2$

Subject to

$20X_1 + 50X_2 \geq 4{,}800$ ……… Phosphate requirement

$80X_1 + 50X_2 \geq 7{,}200$ ………Nitrogen requirement

$X_1, X_2 \geq 0$ ……… Non-negativity restriction

LPP Formulation

Minimize **$Z = 40X_1 + 24X_2$**

Subject to

$20X_1 + 50X_2 \geq 4{,}800$ .... Phosphate requirement

$80X_1 + 50X_2 \geq 7{,}200$ ....Nitrogen requirement

$X_1, X_2 \geq 0$ .... Non-negativity restriction

First introduce some new variables to convert inequalities of the system into equations. The variable required for converting a 'greater than' type of inequality into an equation is called surplus variable and **it represents the excess of what is generated** (given by the LHS of the inequality) **over the requirement** (shown by the RHS value b_i). With surplus variables, S_1 and S_2, respectively, for the first and the second constraints, the augmented problem shall be

Minimize **$Z = 40X_1 + 24X_2 + 0S_1 + 0S_2$**

Subject to

$20X_1 + 50X_2 - S_1 = 4{,}800$

$80X_1 + 50X_2 - S_2 = 7{,}200$

$X_1, X_2, S_1, S_2 \geq 0$

LPP Formulation

Minimize $Z = 40X_1 + 24X_2$

Subject to

$20X_1 + 50X_2 \geq 4,800$ ……… Phosphate requirement

$80X_1 + 50X_2 \geq 7,200$ ………Nitrogen requirement

$X_1, X_2 \geq 0$ ……… Non-negativity restriction

Minimize $Z = 40X_1 + 24X_2 + 0S_1 + 0S_2$

Subject to

$20X_1 + 50X_2 - S_1 = 4,800$

$80X_1 + 50X_2 - S_2 = 7,200$

$X_1, X_2, S_1, S_2 \geq 0$

We may **recall that the simplex method begins with an initial solution obtained by setting each of the decision variables equal to zero.**

Now, if we set X_1 and X_2 equal to zero in this problem, we get $S_1 = -4,800$ and $S_2 = -7,200$, which is not a feasible solution as it violates the non-negativity restriction. In terms of the simplex tableau, when we write all the information, we observe **that we do not get identity because unlike in case of slack variables,** the coefficient values of surplus variables S_1 and S_2 appear as minus one (-1).

BIG-M Method

In a case **where an identity is not obtained**, as in the problem under consideration, a variant of the **simplex method called the Big-M method is employed.** In this method, we **add artificial variables into the model to obtain an initial solution**. However, unlike slack or surplus variables, **artificial variables have no tangible relationship with the decision problem**. Their **sole purpose is to provide an initial solution to the given problem**. When artificial variables are introduced in the problem under consideration, its constraints appear as

$$20X_1 + 50X_2 - S_1 + A_1 = 4,800$$

$$80X_1 + 50\ X_2 - S_2 + A_2 = 7,200$$

BIG-M Method

- It is significant to understand that the artificial variables, which are not seen to disturb the equations already obtained since **they are not 'real',** are introduced for the **limited purpose of obtaining an initial solution** and are required for the constraints of type, or the constraints with `=' sign.
- It is not relevant whether the objective function is of the minimisation or the maximisation type. Obviously, **since artificial variables do not represent any quantity relating to the decision problem**, **they must be driven out of the system and must not show in the final solution** **(and if at all they do, it represents a situation of infeasibility)**. This can be ensured by assigning an extremely high cost to them.
- Generally, **a value M is assigned to each artificial variable**, where **M represents a number higher than any finite number.** For this reason, the method of solving the problems where artificial variables are involved is termed as the Big-M Method.

BIG-M Method

- Thus, when the problem is of the minimisation nature**, we assign in the objective function a coefficient of +M** to each of the artificial variables.
- On the other hand, for the problems with the objective function of maximisation type, each of **the artificial variables introduced has a coefficient –M.**
- Note that it is attempted to prohibit the appearance of artificial variables in the solution by assigning these coefficients: **an extremely large value when the objective is to minimise and an extremely small (negative) value when it is desired to maximise the objective function.**

BIG-M Method

- For our present example, the objective function would appear as

Minimize $Z = 40X_1 + 24X_2 + 0S_1 + 0S_2 + MA_1 + MA_2$

It is to note that the initial solution obtained using artificial variables is not a feasible solution to the given problem. It only gives a starting point and the artificial variables are driven out in the normal course of applying the simplex algorithm. In fact, as long as an artificial variable is included in the basis, the solution is infeasible and only a solution to the problem that does not include an artificial variable in the basis, represents a basic feasible solution to the problem.

Standard Form

Standard form:

Objective function:

Minimize $Z = 40X_1 + 24X_2 + 0S_1 + 0S_2 + MA_1 + MA_2$

Subject to constraints:

$$20X_1 + 50X_2 - S_1 + A_1 = 4{,}800$$

$$80X_1 + 50X_2 - S_2 + A_2 = 7{,}200$$

$X_1, X_2, S_1, S_2\ A_1, A_1 \geq 0$

Initial Basic Feasible Solution: Table 1

Standard form:
Objective function:
Minimize $Z = 40X_1 + 24X_2 + 0S_1 + 0S_2 + MA_1 + MA_2$
Subject to constraints:
$20X_1 + 50X_2 - S_1 + A_1 = 4,800$
$80X_1 + 50X_2 - S_2 + A_2 = 7,200$
$X_1, X_2, S_1, S_2, A_1, A_1 \geq 0$

Step 3: Writing 1st simplex table (Initial Basic Feasible Solution)

- The structure of the simplex table is as given below:

C_j			40	24	0	0	M	M	Replacement Ratio
Basis									
c	x	b	X_1	X_2	S_1	S_2	A_1	A_2	
M	A_1	4800	20	50	-1	0	1	0	4800/50=96
M	A_2	7200	80	50	0	-1	0	1	7200/50=144
Z_j			100M	100M	-M	-M	M	M	
$\Delta = C_j - Z_j$			40 -100M	24 - 100M	M	M	0	0	

Standard form:
Objective function:
Minimize $Z = 40X_1 + 24X_2 + 0S_1 + 0S_2 + MA_1 + MA_2$
Subject to constraints:

$20X_1 + 50X_2 - S_1 + A_1 = 4{,}800$

$80X_1 + 50X_2 - S_2 + A_2 = 7{,}200$

$X_1, X_2, S_1, S_2, A_1, A_1 \geq 0$

Initial Basic Feasible Solution: Table 1

Step 3: Writing 1st simplex table (Initial Basic Feasible Solution)

- The structure of the simplex table is as given below:

C_j			40	24	0	0	M	M	Replacement Ratio
Basis									
c	x	b	X_1	X_2	S_1	S_2	A_1	A_2	
M	A_1	4800	20	50	-1	0	1	0	4800/50=96
M	A_2	7200	80	50	0	-1	0	1	7200/50=144
Z_j			100M	100M	-M	-M	M	M	
$\Delta = C_j - Z_j$			40 -100M	24 - 100M	M	M	0	0	

Standard form:

Objective function:

Minimize $Z = 40X_1 + 24X_2 + 0S_1 + 0S_2 + MA_1 + MA_2$

Subject to constraints:

$20X_1 + 50X_2 - S_1 + A_1 = 4{,}800$

$80X_1 + 50X_2 - S_2 + A_2 = 7{,}200$

$X_1, X_2, S_1, S_2\ A_1, A_1 \geq 0$

Initial Basic Feasible Solution: Table 2

Step 3: Writing 1st simplex table (Initial Basic Feasible Solution)

- The structure of the simplex table is as given below:

C_j			**40**	**24**	**0**	**0**	**M**	**M**	**Replacement Ratio**
Basis									
c	x	b	X_1	X_2	S_1	S_2	A_1	A_2	
24	X_2	96	2/5	1	-1/50	0	1/50	0	
M	A_2	2400	60	0	1	-1	-1	1	
Z_j			48/5+60M	24	-12/25+M	-M	12/25-M	M	
$\Delta = C_j - Z_j$			152/5-60M	0	12/25-M	M	2M-12/25	0	

Initial Basic Feasible Solution: Table 2

Standard form:
Objective function:
Minimize $Z = 40X_1 + 24X_2 + 0S_1 + 0S_2 + MA_1 + MA_2$
Subject to constraints:
$20X_1 + 50X_2 - S_1 + A_1 = 4,800$
$80X_1 + 50X_2 - S_2 + A_2 = 7,200$
$X_1, X_2, S_1, S_2, A_1, A_1 \geq 0$

Step 3: Writing 1st simplex table (Initial Basic Feasible Solution)

- The structure of the simplex table is as given below:

C_j			**40**	**24**	**0**	**0**	**M**	**M**	**Replacement Ratio**
Basis									
c	x	b	X_1	X_2	S_1	S_2	A_1	A_2	
24	X_2	96	2/5	1	-1/50	0	1/50	0	96*5/2=240
M	A_2	2400	60	0	1	-1	-1	1	2400/60=40
Z_j			48/5+60M	24	-12/25+M	-M	12/25-M	M	
$\Delta = C_j - Z_j$			152/5-60M	0	12/25-M	M	2M-12/25	0	

Initial Basic Feasible Solution: Table 3

Standard form:
Objective function:
Minimize $Z = 40X_1 + 24X_2 + 0S_1 + 0S_2 + MA_1 + MA_2$
Subject to constraints:
$20X_1 + 50X_2 - S_1 + A_1 = 4{,}800$
$80X_1 + 50X_2 - S_2 + A_2 = 7{,}200$
$X_1, X_2, S_1, S_2, A_1, A_1 \geq 0$

Step 3: Writing 1st simplex table (Initial Basic Feasible Solution)

- The structure of the simplex table is as given below:

C_j			**40**	**24**	**0**	**0**	**M**	**M**	**Replacement Ratio**
Basis									
c	x	b	X_1	X_2	S_1	S_2	A_1	A_2	
24	X_2	80	0	1	-2/75	1/150	2/75	-1/150	
40	X_1	40	1	0	1/60	-1/60	-1/60	1/60	
Z_j			40	24	2/75	-38/75	-2/75	-38/75	
$\Delta = C_j - Z_j$			0	0	-2/75	38/75	M+2/75	M+38/75	

Initial Basic Feasible Solution: Table 3

Standard form:
Objective function:
Minimize $Z = 40X_1 + 24X_2 + 0S_1 + 0S_2 + MA_1 + MA_2$
Subject to constraints:
$20X_1 + 50X_2 - S_1 + A_1 = 4{,}800$
$80X_1 + 50X_2 - S_2 + A_2 = 7{,}200$
$X_1, X_2, S_1, S_2, A_1, A_1 \geq 0$

Step 3: Writing 1st simplex table (Initial Basic Feasible Solution)

- The structure of the simplex table is as given below:

C_j			**40**	**24**	**0**	**0**	**M**	**M**	**Replacement Ratio**
Basis									
c	x	b	X_1	X_2	S_1	S_2	A_1	A_2	
24	X_2	80	0	1	-2/75	1/150	2/75	-1/150	80/-2/75=-3000
40	X_1	40	1	0	1/60	-1/60	-1/60	1/60	40*60=2400
Z_j			40	24	2/75	-38/75	-2/75	-38/75	
$\Delta = C_j - Z_j$			0	0	-2/75	38/75	M+2/75	M+38/75	

Initial Basic Feasible Solution: Table 4

Standard form:
Objective function:
Minimize $Z = 40X_1 + 24X_2 + 0S_1 + 0S_2 + MA_1 + MA_2$
Subject to constraints:
$20X_1 + 50X_2 - S_1 + A_1 = 4,800$
$80X_1 + 50X_2 - S_2 + A_2 = 7,200$
$X_1, X_2, S_1, S_2, A_1, A_1 \geq 0$

Step 3: Writing 1st simplex table (Initial Basic Feasible Solution)

- The structure of the simplex table is as given below:

C_j			**40**	**24**	**0**	**0**	**M**	**M**	**Replacement Ratio**
Basis									
c	x	b	X_1	X_2	S_1	S_2	A_1	A_2	
24	X_2	144	8/5	1	0	-1/50	0	1/50	
0	S_1	2400	60	0	1	-1	-1	1	
Z_j			192/5	24	0	-12/25	0	12/25	
$\Delta = C_j - Z_j$			8/5	0	0	12/25	M	M-12/25	

Optimal Solution

Standard form:
Objective function:
Minimize $Z = 40X_1 + 24X_2 + 0S_1 + 0S_2 + MA_1 + MA_2$
Subject to constraints:
$20X_1 + 50X_2 - S_1 + A_1 = 4,800$
$80X_1 + 50X_2 - S_2 + A_2 = 7,200$
$X_1, X_2, S_1, S_2, A_1, A_1 \geq 0$

		C_j	**40**	**24**	**0**	**0**	**M**	**M**
		Basis						
c	x	b	X_1	X_2	S_1	S_2	A_1	A_2
24	X_2	144	8/5	1	0	-1/50	0	1/50
0	S_1	2400	60	0	1	-1	-1	1
		Z_j	192/5	24	0	-12/25	0	12/25
		$\Delta = C_j - Z_j$	8/5	0	0	12/25	M	M-12/25

All $C_j - Z_j$; values are either zero or positive. There is no negative **Δ**. Hence, solution is optimal.

According to the optimal solution, the objective function value is

Z = 40*0+24*144+0*2400+0*0+0*M+0*M

Z =Rs.3456

The value of S_1= 2400 indicates the phosphate ingredient obtained by buying the least cost mix.

Simplex Method

The Maximization Case:

Example 3:

A firm is engaged in producing two products, A and B. Each unit of product A requires two kg of raw material and four labour hours for processing, whereas each unit of product B requires three kg of raw material and three hours of labour, of the same type. Every week, the firm has an availability of 60 kg of raw material and 96 labour hours. One unit of product A sold yields Rs.40 and one unit of product B sold gives Rs.35 as profit.

Formulate this problem as a linear programming problem to determine as to how many units of each of the products should be produced per week so that the firm can earn the maximum profit. Assume that there is no marketing constraint so that all that is produced can be sold.

Simplex Method

The Maximization Case:

Example 3:

A firm is engaged in producing two products, A and B. Each unit of product A requires two kg of raw material and four labour hours for processing, whereas each unit of product B requires three kg of raw material and three hours of labour, of the same type. Every week, the firm has an availability of 60 kg of raw material and 96 labour hours. One unit of product A sold yields Rs.40 and one unit of product B sold gives Rs.35 as profit.

Formulate this problem as a linear programming problem to determine as to how many units of each of the products should be produced per week so that the firm can earn the maximum profit. Assume that there is no marketing constraint so that all that is produced can be sold.

Maximise $Z = 40X_1 + 35X_2$ Profit

Subject to

$2X_1 + 3X_2 \leq 60$ Raw material constraint

$4X_1 + 3X_2 \leq 96$ Labour hours constraint

$X_1, X_2 \geq 0$ Non-negativity restriction

Simplex Method

The Maximization Case:

Original LPP

Maximise $Z = 40X_1 + 35X_2$ Profit

Subject to

$2X_1 + 3X_2 \leq 60$ Raw material constraint

$4X_1 + 3X_2 \leq 96$ Labour hours constraint

$X_1, X_2 \geq 0$ Non-negativity restriction

Thus, when slack variables have been introduced in both the constraints, the original LP model can now be replaced with an equivalent problem. Both of these are shown here:

Augmented LPP

Maximize $Z = 40X_1 + 35X_2 + 0S_1 + 0S_2$

Subject to

$2X_1 + 3X_2 + 0S_1 = 60$ Raw material constraint

$4X_1 + 3X_2 + 0S_2 = 96$ Labour hours constraint

$X_1, X_2, S_1, S_2 \geq 0$ Non-negativity restriction

Augmented LPP

Maximize $Z = 40X_1 + 35X_2 + 0S_1 + 0S_2$

Subject to

$2X_1 + 3X_2 + 0S_1 = 60$ Raw material constraint

$4X_1 + 3X_2 + 0S_2 = 96$ Labour hours constraint

$X_1, X_2, S_1, S_2 \geq 0$ Non-negativity restriction

asic Feasible Solution

Step 3: Writing 1st simplex table (Initial Basic Feasible Solution)

- The structure of the simplex table is as given below:

C_j							Replacement Ratio
Basis							
c	x	b	X_1	X_2	S_1	S_2	
Z_j							
$\Delta = C_j - Z_j$							

Augmented LPP

Maximize $Z = 40X_1 + 35X_2 + 0S_1 + 0S_2$

Subject to

$2X_1 + 3X_2 + 0S_1 = 60$ Raw material constraint

$4X_1 + 3X_2 + 0S_2 = 96$ Labour hours constraint

$X_1, X_2, S_1, S_2 \geq 0$ Non-negativity restriction

asic Feasible Solution

Step 3: Writing 1st simplex table (Initial Basic Feasible Solution)

- The structure of the simplex table is as given below:

C_j			40	35	0	0	Replacement Ratio
Basis							
c	x	b	X_1	X_2	S_1	S_2	
0	S_1	60	2	3	1	0	
0	S_2	96	4	3	0	1	
Z_j			0	0	0	0	
$\Delta = C_j - Z_j$			40	35	0	0	

asic Feasible Solution

Augmented LPP

Maximize $Z = 40X_1 + 35X_2 + 0S_1 + 0S_2$

Subject to

$2X_1 + 3X_2 + 0S_1 = 60$ Raw material constraint

$4X_1 + 3X_2 + 0S_2 = 96$ Labour hours constraint

$X_1, X_2, S_1, S_2 \geq 0$ Non-negativity restriction

Step 3: Writing 1st simplex table (Initial Basic Feasible Solution)

- The structure of the simplex table is as given below:

C_j			40	35	0	0	Replacement Ratio
Basis							
c	x	b	X_1	X_2	S_1	S_2	
0	S_1	60	2	3	1	0	60/2=30
0	S_2	96	4	3	0	1	96/4=24
Z_j			0	0	0	0	
$\Delta = C_j - Z_j$			40	35	0	0	

asic Feasible Solution

Augmented LPP

Maximize $Z = 40X_1 + 35X_2 + 0S_1 + 0S_2$

Subject to

$2X_1 + 3X_2 + 0S_1 = 60$ Raw material constraint

$4X_1 + 3X_2 + 0S_2 = 96$ Labour hours constraint

$X_1, X_2, S_1, S_2 \geq 0$ Non-negativity restriction

Step 3: Writing 1st simplex table (Initial Basic Feasible Solution)

- The structure of the simplex table is as given below:

C_j			40	35	0	0	Replacement Ratio
Basis							
c	x	b	X_1	X_2	S_1	S_2	
0	S_1	12	0	3/2	1	-1/2	
40	X_1	24	1	¾	0	¼	
Z_j			40	30	0	10	
$\Delta = C_j - Z_j$			0	5	0	-10	

Augmented LPP

Maximize Z = $40X_1 + 35X_2 + 0S_1 + 0S_2$

Subject to

$2X_1 + 3X_2 + 0S_1 = 60$ Raw material constraint

$4X_1 + 3X_2 + 0S_2 = 96$ Labour hours constraint

$X_1, X_2, S_1, S_2 \geq 0$ Non-negativity restriction

asic Feasible Solution

Step 3: Writing 1st simplex table (Initial Basic Feasible Solution)

- The structure of the simplex table is as given below:

C_j			40	35	0	0	Replacement Ratio
Basis							
c	x	b	X_1	X_2	S_1	S_2	
0	S_1	12	0	3/2	1	-1/2	12*2/3=8
40	X_1	24	1	¾	0	¼	24*4/3=32
Z_j			40	30	0	10	
$\Delta = C_j - Z_j$			0	5	0	-10	

Augmented LPP

Maximize $Z = 40X_1 + 35X_2 + 0S_1 + 0S_2$

Subject to

$2X_1 + 3X_2 + 0S_1 = 60$ Raw material constraint

$4X_1 + 3X_2 + 0S_2 = 96$ Labour hours constraint

$X_1, X_2, S_1, S_2 \geq 0$ Non-negativity restriction

asic Feasible Solution

Step 3: Writing 1st simplex table (Initial Basic Feasible Solution)

- The structure of the simplex table is as given below:

C_j			40	35	0	0	Replacement Ratio
Basis							
c	x	b	X_1	X_2	S_1	S_2	
35	X_2	8	0	1	2/3	-1/3	
40	X_1	18	1	0	-1/2	1/2	
Z_j			40	35	10/3	25/3	
$\Delta = C_j - Z_j$			0	0	-10/3	-25/3	

Optimal Solution

Augmented LPP

Maximize $Z = 40X_1 + 35X_2 + 0S_1 + 0S_2$

Subject to

$2X_1 + 3X_2 + 0S_1 = 60$ Raw material constraint

$4X_1 + 3X_2 + 0S_2 = 96$ Labour hours constraint

$X_1, X_2, S_1, S_2 \geq 0$ Non-negativity restriction

C_j			40	35	0	0
Basis						
c	x	b	X_1	X_2	S_1	S_2
35	X_2	8	0	1	2/3	-1/3
40	X_1	18	1	0	-1/2	1/2
Z_j			40	35	10/3	25/3
$\Delta = C_j - Z_j$			0	0	-10/3	-25/3

All $C_j - Z_j$; values are either zero or negative. There is no positive **Δ**. Hence, solution is optimal.

According to the optimal solution, the objective function value is

Z = 40*18+35*8+0*0+0*0

Z =Rs.1000

By this, both the resources would be fully utilized.

A Mixed Constraint Problem

A mixed constraint problem includes a combination of ≤, =, and ≥ constraints.

Example 4:

A leather shop makes custom-designed, hand-tooled briefcases and luggage. The shop makes a $400 profit from each briefcase and a $200 profit from each piece of luggage. (The profit for briefcases is higher because briefcases require more hand tooling.) The shop has a contract to provide a store with exactly 30 items per month. A tannery supplies the shop with at least 80 square yards of leather per month. The shop must use at least this amount but can order more. Each briefcase requires 2 square yards of leather; each piece of luggage requires 8 square yards of leather. From past performance, the shop owners know they cannot make more than 20 briefcases per month. They want to know the number of briefcases and pieces of luggage to produce in order to maximize profit.

A Mixed Constraint Problem

A leather shop makes custom-designed, hand-tooled briefcases and luggage. The shop makes a $400 profit from each briefcase and a $200 profit from each piece of luggage. (The profit for briefcases is higher because briefcases require more hand tooling.) The shop has a contract to provide a store with exactly 30 items per month. A tannery supplies the shop with at least 80 square yards of leather per month. The shop must use at least this amount but can order more. Each briefcase requires 2 square yards of leather; each piece of luggage requires 8 square yards of leather. From past performance, the shop owners know they cannot make more than 20 briefcases per month. They want to know the number of briefcases and pieces of luggage to produce in order to maximize profit.

Max. $Z = 400X_1 + 200X_2$

Subject to constraints:

$X_1 + X_2 = 30$ contracted items

$2X_1 + 8X_2 \geq 80$ square yards of leather

$X_1 \leq 20$ briefcases

$X_1, X_2, \geq 0$

Decision Variable

X_1 = briefcases

X_2 = pieces of luggage

A Mixed Constraint Problem

- The first step in the simplex method is to transform the inequalities into equations.
- The first constraint for the contracted items is already an equation; therefore, it is not necessary to add a slack variable.
- There can be no slack in the contract with the store because exactly 30 items must be delivered.
- Even though this equation already appears to be in the necessary form for simplex solution, let us test it at the origin to see if it meets the starting requirements:

$X_1 + X_2 = 30$

$0 + 0 = 30$

$0 \neq 30$

- Because zero does not equal 30, the constraint is not feasible in this form.
- Recall that a constraint did not work at the origin either in an earlier problem.
- Therefore, an artificial variable was added. The same thing can be done here:

$X_1 + X_2 + A_1 = 30$

Now at the origin, where X_1=0 and X_2=0

$0 + 0 + A_1 = 30$

$A_1 = 30$

A Mixed Constraint Problem

- The constraint for leather is a inequality. It is converted to equation form by subtracting a surplus variable and adding an artificial variable:

$$2X_1 + 8X_2 - S_1 + A_2 = 80$$

- The final constraint is a ≤ inequality and is transformed by adding a slack variable:

$$X_1 + S_2 = 20$$

A Mixed Constraint Problem

- Any time a constraint is initially an equation, an artificial variable is added. However, the artificial variable cannot be assigned a value of M in the objective function of a maximization problem. Because the objective is to maximize profit, a positive M value would represent a large positive profit that would definitely end up in the final solution. Because an artificial variable has no real meaning and is inserted into the model merely to create an initial solution at the origin, its existence in the final solution would render the solution meaningless. To prevent this from happening, we must give the artificial variable a large cost contribution, or –M.

The completely transformed linear programming problem is as follows:

Max. $Z = 400X_1 + 200X_2 + 0S_1 + 0S_2 - MA_1 - MA_2$

Subject to constraints:

$X_1 + X_2 + A_1 = 30$

$2X_1 + 8X_2 - S_1 + A_2 = 80$

$X_1 + S_2 = 20$

$X_1, X_2, S_1, S_2, A_1, A_2 \geq 0$

The Initial Simplex Tableau

Subject to constraints:

$X_1 + X_2 + A_1 = 30$

$2X_1 + 8X_2 - S_1 + A_2 = 80$

$X_1 + S_2 = 20$

$X_1, X_2, S_1, S_2, A_1, A_2 \geq 0$

- The initial simplex tableau for this model is shown in Table 1. Notice that the basic solution variables are a mix of artificial and slack variables. Note also that the third-row quotient for determining the pivot row (20/0) is an undefined value, or ∞. Therefore, this row would never be considered as a candidate for the pivot row. The second, third, and optimal tableaus for this problem are shown in Tables 2, 3, and 4.

C_j			**400**	**200**	**0**	**0**	**-M**	**-M**	**Replacement Ratio**
Basis									
c	x	b	X_1	X_2	S_1	S_2	A_1	A_2	
-M	A_1	30	1	1	0	0	1	0	
-M	A_2	80	2	8	-1	0	0	1	
0	S_2	20	1	0	0	1	0	0	
Z_j			-3M	-9M	M	0	-M	-M	
$\Delta = C_j - Z_j$			400 -3M	200 + 9M	-M	0	0	0	

The Second Simplex Tableau

Subject to constraints:

$X_1 + X_2 + A_1 = 30$

$2X_1 + 8X_2 – S_1 + A_2 = 80$

$X_1 + S_2 = 20$

$X_1, X_2, S_1, S_2, A_1, A_2 \geq 0$

C_j			400	200	0	0	-M	-M	Replacement Ratio
Basis									
c	x	b	X_1	X_2	S_1	S_2	A_1	A_2	
-M	A_1	30	1	1	0	0	1	0	30/1=30
-M	A_2	80	2	8	-1	0	0	1	80/8=10
0	S_2	20	1	0	0	1	0	0	20/0=20
Z_j			-3M	-9M	M	0	-M	-M	
$\Delta = C_j - Z_j$			400 -3M	200 + 9M	-M	0	0	0	

The Second Simplex Tableau

Subject to constraints:

$X_1 + X_2 + A_1 = 30$

$2X_1 + 8X_2 - S_1 + A_2 = 80$

$X_1 + S_2 = 20$

$X_1, X_2, S_1, S_2, A_1, A_2 \geq 0$

C_j			400	200	0	0	-M	Replacement Ratio
Basis								
c	x	b	X_1	X_2	S_1	S_2	A_1	
-M	A_1	20	3/4	0	1/8	0	1	20*4/3=80/3=26.6
200	X_2	10	1/4	1	-1/8	0	0	10*4=40
0	S_2	20	1	0	0	1	0	20/1=20
Z_j			50-3M/4	200	-25-M/8	0	-M	
$\Delta = C_j - Z_j$			350+3M/4	0	25+M/8	0	0	

The Third Simplex Tableau

Subject to constraints:

$X_1 + X_2 + A_1 = 30$

$2X_1 + 8X_2 – S_1 + A_2 = 80$

$X_1 + S_2 = 20$

$X_1, X_2, S_1, S_2, A_1, A_2 \geq 0$

C_j			**400**	**200**	**0**	**0**	**-M**
Basis							
c	x	b	X_1	X_2	S_1	S_2	A_1
-M	A_1	5	0	0	1/8	-3/4	1
200	X_2	5	0	1	-1/8	-1/4	0
400	X_1	20	1	0	0	1	0
Z_j			400	200	-25-M/8	350+3M/4	-M
$\Delta = C_j - Z_j$			0	0	25+M/8	-350-3M/4	0

The Third Simplex Tableau

Subject to constraints:

$X_1 + X_2 + A_1 = 30$

$2X_1 + 8X_2 – S_1 + A_2 = 80$

$X_1 + S_2 = 20$

$X_1, X_2, S_1, S_2, A_1, A_2 \geq 0$

Note also that the third-row quotient for determining the pivot row (20/0) is an undefined value, or ∞.

C_j			400	200	0	0	-M	Replacement Ratio
Basis								
c	x	b	X_1	X_2	S_1	S_2	A_1	
-M	A_1	5	0	0	1/8	-3/4	1	5*8=40
200	X_2	5	0	1	-1/8	-1/4	0	5*-8=-40
400	X_1	20	1	0	0	1	0	20/0=20
Z_j			400	200	-25-M/8	350+3M/4	-M	
$\Delta = C_j - Z_j$			0	0	25+M/8	-350-3M/4	0	

The Optimal Simplex Tableau

Subject to constraints:

$X_1 + X_2 + A_1 = 30$

$2X_1 + 8X_2 – S_1 + A_2 = 80$

$X_1 + S_2 = 20$

$X_1, X_2 , S_1 , S_2 , A_1 , A_2 \geq 0$

Note also that the third-row quotient for determining the pivot row (20/0) is an undefined value, or ∞.

C_j			400	200	0	0
Basis						
c	x	b	X_1	X_2	S_1	S_2
0	S_1	40	0	0	1	-6
200	X_2	10	0	1	0	-1
400	X_1	20	1	0	0	1
Z_j			400	200	0	200
$\Delta = C_j - Z_j$			0	0	0	-200

The Optimal Simplex Tableau

The completely transformed linear programming problem is as follows:

Max. $Z = 400X_1 + 200X_2 + 0S_1 + 0S_2 - MA_1 - MA_2$

C_j			400	200	0	0
Basis						
c	x	b	X_1	X_2	S_1	S_2
0	S_1	40	0	0	1	-6
200	X_2	10	0	1	0	-1
400	X_1	20	1	0	0	1
Z_j			400	200	0	200
$\Delta = C_j - Z_j$			0	0	0	-200

The solution for the leather shop problem is

X_1 = 20 briefcases

X_2 = 10 pieces of luggage

S_1 = 40 extra square yards of leather

The completely transformed linear programming problem is as follows:

Max. Z = 400*20 + 200*10 = 10,000 profit per month

The Optimal Simplex Tableau

It is now possible to summarize a set of rules for transforming all three types of model constraints:

		Objective Function Coefficient	
Constraint	**Adjustment**	**Maximization**	**Minimization**
$\leq$	Add a slack variable	0	0
$=$	Add an artificial variable	$-M$	M
$\geq$	Subtract a surplus variable	0	0
	and add an artificial variable	$-M$	M

Schematic Presentation of Simplex Method